J.R. INSTITUTE OF MATHEMATICS

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Chapter - 1

Introduction : The subject of differential equations constitutes a large and very important branch of modern mathematics. From the early days of the calculus the subject has been an area of great theoretical research and practical applications and it continues to be so in our day. This much stated, several questions naturally arise. Just what does it signify where and how do differential equations originate and of what use are they ? Confronted with a differential equation, What does one do with it, how does one do it and what are the results of such activity ? These questions indicate three major aspects of the subject : theory, method and application. The purpose of this chapter is to introduce the reader to the basic aspects of the subject and at the same time give a brief survey of the three aspects just mentioned.

1. Differential Equations and Their classification :

Definition : An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation.

e.g., $$\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0 \quad \text{.....(1)}$$

$$\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t \quad \text{.....(2)}$$

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v \quad \text{.....(3)}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{.....(4)}$$

clearly some kind of classification must be made. To begin with, we classify differential equations according to whether there is one or more than one independent variable involved.

Def. Ordinary differential equation : A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation.

Equations (1) and (2) are ordinary differential equations.

Def. Partial differential equation : A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variable is called a partial differential equation.

Equations (3) and (4) are partial differential equations.

Order of a differential equation : The order of the highest ordered derivatives involved in a differential equation is called the order of the differential equation. The ordinary differential equation (1) is of second order. Equation (2) is an ordinary differential equation of fourth order.

Degree of a differential equation : The degree of a differential equation is the degree of the highest order differential derivative, after the equation has been made free of radicals and fractions as far as the derivatives are concerned.

e.g., consider the differential equation $\frac{d^2y}{dx^2} = c\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

To find the degree of differential equation, we have to make it free from square root. So squaring both sides, we get $\left(\frac{d^2y}{dx^2}\right)^2 = c^2\left[1+\left(\frac{dy}{dx}\right)^2\right]$

Here, the degree of highest derivative is 2. So degree of differential equation is 2.

Def. Linear differential equation : A linear differential equation of order *n*, in the dependent variable *y* and the independent variable *x*, is an equation that is in, or can be expressed in, the form

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = b(x)$$

Where a_0 is not identically zero. Observe that the dependent variable and its various derivatives occur to the first degree only and that no product of *y* and/or any of its derivatives are present, and that no transcendental functions of *y* and /or its derivatives occur.

e.g., $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$ and $\frac{d^4y}{dx^4} + x^2\frac{d^3y}{dx^3} + x^3\frac{dy}{dx} = xe^x$

both are linear differential equations.

Def. Non linear ordinary differential equation : A non linear ordinary differential equation is an ordinary differential equation that is not linear.

e.g., $$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y^2 = 0,$$

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 + 6y = 0,$$

$$\frac{d^2y}{dx^2} + 5y\frac{dy}{dx} + 6y = 0$$

all are non-linear differential equations.

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Formation of differential equations : The differential equation can be obtained by differentiating an ordinary equation and eliminating the arbitrary constants among them.

Example 1. Find the differential equation of all straight lines in a plane.

Solution : Consider the equation of a plane is $ax+by=d$.

Thus the straight line in in *xy*-plane is given by $ax+by=d$ ……(1)

where a,b,d are arbitrary constants.

On differentiating (1) with respect to '*x*', we get

$$a+b\frac{dy}{dx}=0$$

Again differentiating with respect to '*x*', we get

$$b\frac{d^2y}{dx^2}=0 \quad\Rightarrow\quad \frac{d^2y}{dx^2}=0,$$

which is a differential equation of order 2 and degree 1.

Example 2. Find the differential equation of all circles of radius '*a*' whose centres lie on the *y*-axis.

Solution : Let the centre of any such circle is $(0,k)$, then the equation of the circle is

$$(x-0)^2+(y-k)^2=a^2 \quad\Rightarrow\quad x^2+(y-k)^2=a^2 \qquad \text{...... (1)}$$

where *k* is the arbitrary constant. Differentiating (1) with respect to '*x*', we have

$$2x+2(y-k)\frac{dy}{dx}=0 \quad\Rightarrow\quad (y-k)=\frac{-x}{\left(\frac{dy}{dx}\right)}$$

Substituting the value of $(y-k)$ in (1), we obtain

$$x^2+\frac{x^2}{\left(\frac{dy}{dx}\right)^2}=a^2 \quad\Rightarrow\quad x^2\left[1+\left(\frac{dy}{dx}\right)^2\right]=a^2$$

which is the required differential equation.

Example 3. Find the differential equation of all parabolas whose vertex is (h,k).

Solution : The equation of the parabolas is $(y-k)^2=4a(x-h)$ ……(1)

where *a* is a constant. Differentiating (1) with respect to *x*, we get

$$2(y-k)\frac{dy}{dx}=4a$$

Hence from (1), we have

$$(y-k)^2 = 2(y-k)\frac{dy}{dx}(x-h) \qquad \Rightarrow \qquad y-k = 2(x-h)\frac{dy}{dx}$$

which is required differential equation.

Solution of differential equation : Consider the n^{th} order ordinary differential equation

$$F\left[x, y, \frac{dy}{dx}, \ldots., \frac{d^n y}{dx^n}\right] = 0 \qquad \ldots.(*)$$

Where F is a real function of its $(n+2)$ arguments $x, y, \frac{dy}{dx}, \ldots., \frac{d^n y}{dx^n}$.

1. Let f be a real function defined and for all x in a real interval I and having an n^{th} derivative (and hence also all lower derivatives) for all $x \in I$. The function f is called an explicit solution of the differential equation (*) on I if it fulfills the following two requirements :

   $$F\left[x, f(x), f'(x), \ldots., f^{(n)}(x)\right] \qquad \ldots.(A)$$

   defined for all $x \in I$, and

   $$F\left[x, f(x), f'(x), \ldots., f^{(n)}(x)\right] = 0 \qquad \ldots.(B)$$

   for all $x \in I$. That is, the substitution of $f(x)$ and its various derivations of y and its corresponding
   derivatives, respectively, in (*) reduces (*) to an identity on I.
2. A relation $g(x, y) = 0$ is called an implicit solution of (*) if this relation defines at least one real function f of the variable x on an interval I such that this function is an explicit solution of (*) on this interval.
3. Both explicit solutions and implicit solutions will usually be called simply solutions. Roughly speaking, then, we may say that a solution of the differential equation (*) is relation-explicit or implicit between x and y, not containing derivatives, which identically satisfies (*). e.g., The function f defined for all real x by

   $$f(x) = 2\sin x + 3\cos x$$

   is an explicit solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$ for all real x.

   Also, the relation $x^2 + y^2 - 25 = 0$ is an implicit solution of the differential equation $x + y\frac{dy}{dx} = 0$ on the interval I defined by $-5 < x < 5$

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Example 1 : Consider the first order differential equation $\frac{dy}{dx} = 2x$ …..(A)

The function f_0 defined for all *s* and *x* by $f_0(x) = x^2$ is a solution of this equation. So also are the functions f_1, f_2 and f_3 defined for all real x by $f_1(x) = x^2 + 1,\ f_2(x) = x^2 + 2$ and $f_3(x) = x^2 + 3$, respectively. In fact, for each real number *c*, the function be defined for all real *x* by

$$f_c(x) = x^2 + c \qquad \text{….(B)}$$

is a solution of the differential equation (A). In other words, the formula (B) defines an infinite family of functions one for each real constant *c*, and every function of this family is a solution of (A). We call the constant *c* in (B) an arbitrary constant or parameter and refer to the family of functions defined by (B) as a one-parameter family of solutions of the differential equation (A). We write this one-parameter family of solutions as $y = x^2 + c$.

Def. Geometric significance of differential equations : A real function *F* may be represented geometrically by a curve $y = F(x)$ in the *xy*-plane and that the value of the derivative of *F* at *x*, $F'(x)$, may be interpreted as the slope of the curve $y = F(x)$ at *x*. Thus the general first-order differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{…..(*)}$$

Where f is a real function, may be interpreted geometrically as defining a slope $f(x, y)$ at every point (x, y) at which the function f is defined. Now assume that the differential equation (*) has a so-called one-parameter family of solutions that can be written in the form

$$y = F(x, c) \qquad \text{….(**)}$$

Where *c* is arbitrary constant or parameter of the family. The one-parameter family of functions defined by (**) is represented geometrically by a so-called one-parameter family of curves in the *xy*-plane, the slope of which are given by the differential equation (*). These curves, the graphs of the solutions of the differential equation (*), are called the integral curves of the differential equation.

Solution of Differential equation : Any relation between dependent and independent variables not containing the derivatives of dependent variable with respect to the independent variable, which on substitution in the differential equation reduces it to an identity, is called a solution of differential equation. A solution or integral is also known as a primitive because the differential equation can be derived from it.

Classification of solution of the differential equation : Let $F\left(x, y, \frac{dy}{dx}, ..., \frac{d^n y}{dx^n}\right) = 0$ be the n^{th} order differential equation.

I. General solution : The general solution of the differential equation is the solution which contains the number of arbitrary constants equal to the order of the differential equation.

e.g., consider the differential equation $\frac{d^2 y}{dx^2} = 9y$, then $y = Ae^{3x} + Be^{-3x}$ is a general solution of this differential equation.

II. Particular solution : If the arbitrary constants in the general solution are assigned specific values, the resultant solution is called a particular solution.

e.g., $y = e^{3x} + 5e^{-3x}$ is a particular solution of the differential equation $\frac{d^2 y}{dx^2} = 9y$.

III. Singular solution : A singular solution is a solution of the differential equation which does not contain any arbitrary constant but can not be derived from general solution. In other words, it is not obtained just by providing some particular values to arbitrary constants in general solution.

Exercise

1. Classify each of the following differential equations as ordinary or partial differential equations ; state the order of each equation ; and determine whether the equation under consideration is linear or non-linear.

   (i) $\frac{dy}{dx} + x^2 y = xe^x$ (ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (iii) $\frac{d^4 y}{dx^4} + 3\left(\frac{d^2 y}{dx^2}\right)^5 + 5y = 0$

2. Show that every function f defined by $f(x) = (x^3 + c)e^{-3x}$ where c is an arbitrary constant, is a solution of the differential equation $\frac{dy}{dx} + 3y = 3x^2 e^{-3x}$.

3. For certain values of the constant m the function f defined by $f(x) = e^{mx}$ is a solution of the differential equation $\frac{d^3 y}{dx^3} - 3\frac{d^2 y}{dx^2} - 4\frac{dy}{dx} + 12y = 0$. Determine all such values of m.

Answers

1. (i) ordinary ; first ; linear (ii) Partial ; second ; linear (iii) ordinary ; fourth ; non-linear
3. 2, 3, –2

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Assignment-1

--------------------- S C Q ---------------------

1. Consider the differential equation

$$\frac{d^2y}{dx^2}+y\sin x=0 \text{ is}$$

   1. a linear ordinary differential equation
   2. a non linear ordinary differential equation
   3. a partial differential equation
   4. none of these

2. Consider the differential equation

$$\frac{d^6x}{dt^6}+\left(\frac{d^4x}{dt^4}\right)\left(\frac{d^3x}{dt^3}\right)+x=t \text{ is}$$

   1. linear ODE
   2. a non linear ODE
   3. a partial ODE
   4. of degree two

3. Consider the differential equation

$$\left(\frac{dr}{ds}\right)^3=\sqrt{\frac{d^2r}{ds^2}+1} \text{ is}$$

   1. a second order linear ODE
   2. a second order non linear ODE
   3. a first order ODE
   4. none of these

4. The order of the differential equation whose general solution is

$$y=c_1\cos 2x+c_2\cos^2 x+c_3\sin^2 x+c_4 \text{ is}$$

   1. 2 2. 3
   3. 4 4. 1

5. The degree of the equation

$$\left(\frac{d^3y}{dx^3}\right)^{\frac{2}{3}}+\left(\frac{d^3y}{dx^3}\right)^{\frac{3}{2}}=0 \text{ is}$$

   1. $\frac{3}{2}$ 2. 5
   3. 4 4. 9

6. The order and degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^{\frac{1}{5}}+\left(\frac{d^4y}{dx^4}\right)y^5+4e^x\cos(2x+5)=0$$

   are respectively
   1. 4, 10 2. 4, 5
   3. 5, 4 4. 10, 4

7. If $y=y(x)$. Then the differential equation

$$y'=\sqrt{x}+\sqrt{y} \text{ is}$$

   1. linear differential equation of order 1
   2. non linear differential equation of order1
   3. a PDE
   4. a PDE of order 1

8. The degree of the differential equation

$$(y'')^2+(y')^2=x\sin\left(\frac{d^2y}{dx^2}\right) \text{ is}$$

   1. 1 2. 2
   3. 3 4. Not defined

9. Which of the following differential equation is linear if $y=y(x)$?

   1. $\frac{dy}{dx}+x^2y=\sqrt{y}$
   2. $\frac{dy}{dx}-\sin x=x^ny$

3. $(1+y)\frac{dy}{dx}+P(x)y=Q(x)$

4. $\sqrt{y+\frac{dy}{dx}}=x+y$

10. If $y=y(x)$ then the differential equation $\frac{dy}{dx}+Py=Q$, is a linear equation of first order if,

1. P is a constant but Q is a function of y.
2. P and Q are functions of y.
3. P is a function of y but Q is a constant.
4. P and Q are functions of x or constants.

-------------------------- M C Q ---------------------

1. Consider the differential equation

$$\left(\frac{d^4y}{dx^4}\right)^{\frac{2}{3}}+\left(\frac{d^4y}{dx^4}\right)^{\frac{3}{2}}=0\text{, then}$$

1. order is 4
2. order is 3
3. degree is 4
4. degree is 9

2. Consider the differential equation

$$\frac{d^4y}{dx^4}+\frac{d^3y}{dx^3}+x\frac{dy}{dx}+y^2=0 \text{ is}$$

1. linear ODE
2. non linear ODE
3. order is 4
4. degree is 1

Answers

----------------------- S C Q --------------------

1. 1	2. 2	3. 2	4. 3
5. 4	6. 2	7. 2	8. 4
9. 2	10. 4		

---------------------- M C Q ---------------------

1. 1,4   2. 2,3,4

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Chapter - 2

First order first degree differential equation :

2.1 Standard form of first order differential equation : The first order differential equation may be expressed in either the derivative form $\frac{dy}{dx} = f(x,y)$ ……(1)

or the differential form $M(x,y)dx + N(x,y)dy = 0$ ……(2)

An equation in one of these forms may readily be written in the other form. e.g., the equation $\frac{dy}{dx} = \frac{x^2+y^2}{x-y}$ is of the form (1). It may be written $(x^2+y^2)dx + (y-x)dy = 0$ which is of the form (2)

2.2 Exact differential equations :

Def. Total differential of a function : Let F be a function of two real variables such that F has continuous first partial derivatives in a domain D. The total differential dF of the function F is defined by the formual $dF(x,y) = \frac{\partial F(x,y)}{\partial x}dx + \frac{\partial F(x,y)}{\partial y}dy$ for all $(x,y) \in D$. e.g., let F be the function of two variables defined by $F(x,y) = xy^2 + 2x^3y$ for all real (x,y). Then $\frac{\partial F(x,y)}{\partial x} = y^2 + 6x^2y$, $\frac{\partial F(x,y)}{\partial y} = 2xy + 2x^3$ and the total differential dF is defined by

$dF(x,y) = (y^2 + 6x^2y)dx + (2xy + 2x^3)dy$ for all real (x,y).

Def. Exact differential equation : The expression $M(x,y)dx + N(x,y)dy = 0$ …..(1)

is called an exact differential in a domain D if there exists a function F of two real variables such that this expression equals the total differential $dF(x,y)$ for all $(x,y) \in D$. That is, expression (1) is an exact differential in D if there exists a function F such that $\frac{\partial F(x,y)}{\partial x} = M(x,y)$, $\frac{\partial F(x,y)}{\partial y} = N(x,y)$ for all $(x,y) \in D$.

If $M(x,y)dx + N(x,y)dy$ is an exact differential, then the differential equation $M(x,y)dx + N(x,y)dy = 0$ is called an exact differential equation. e.g., the differential equation

$$y^2dx + 2xy\,dy = 0 \quad …..(2)$$

is an exact differential equation, since the expression $y^2dx + 2xy\,dy$ is an exact differential. It is the total differential of the function F defined for all (x,y) by $F(x,y) = xy^2$. On the other hand, the more

simple appearing equation $\quad y dx + 2x\, dy = 0 \quad$ …..(3)

obtained from (2) by dividing through by y, is not exact. In case of equation (2), we verified our assertion by actually exhibiting the function F of which the expression $y^2 dx + 2xy\, dy$ is the total differential. But in the case of equation (3), we did not backup our statement by showing that there is no function F such that $ydx + 2xdy$ is its total differential. It is clear that we need a simple test to determine whether or not a given differential equation is exact. This is given by the following theorem.

Theorem : Consider the differential equation $\quad M(x, y)\, dx + N(x, y)\, dy = 0 \quad$ …..(1)

Where M and N have continuous first order partial derivatives at all points (x, y) in a rectangular domain D. Then

1. If the differential equation (1) is exact in D, then $\dfrac{\partial M(x,y)}{\partial y} = \dfrac{\partial N(x,y)}{\partial x}$ for all $(x, y) \in D$
2. Conversely, if $\dfrac{\partial M(x,y)}{\partial y} = \dfrac{\partial N(x,y)}{\partial x}$ for all $(x, y) \in D$, then the differential equation (1) is exact in D.

Solution of an exact differential equation : Suppose the differential equation $M(x, y)\, dx + N(x, y)\, dy = 0$ is an exact differential equation. Then solution of this equation is given by

$$\int\limits_{y\,\text{constant}} M\, dx + \int (\text{terms in } N \text{ not containing } x)\, dy = c$$ where c is an arbitrary constant.

Steps for solving exact differential equations :

Step I. Integrate M with respect to x keeping y constant.

Step II. Integrate only those terms of N which do not contain x, with respect to y.

Step III. Result of Step I + Result of Step II = constant, is the solution of the given differential equation.

Integrating factor : Given the differential equation $M(x, y)\, dx + N(x, y)\, dy = 0$ if $\dfrac{\partial M(x,y)}{\partial y} = \dfrac{\partial N(x,y)}{\partial x}$ then equation is exact and we can obtain a one-parameter family of solutions by one of the procedures explained above. But if $\dfrac{\partial M(x,y)}{\partial y} \neq \dfrac{\partial N(x,y)}{\partial x}$ then the equation is not exact and the above procedures do not apply. What shall we do in such a case ? Perhaps we can multiply the non exact equation by some expression that will transform it into an essentially equivalent exact equation. If so, we can proceed to solve the resulting exact equation by one of the above procedures. Let us consider again the equation $\quad y\, dx + 2x\, dy = 0 \quad$ …...(1)

is not exact. However, if we multiply equation (1) by y, it is transformed into the essentially equivalent equation $y^2\,dx + 2xy\,dy = 0$ .....(2)

which is exact. Since this resulting exact equation (2) is integrable, we call y as integrating factor of equation (1). In general we have the following definition :

Def. Integrating factor of the differential equation : If the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0 \qquad .....(*)$$

is not exact in a domain D but the differential equation $u(x, y)\,M(x, y)\,dx + u(x, y)\,N(x, y)\,dy = 0$ is exact in D, then $u(x, y)$ is called an integrating factor of the differential equation (*).

Note : There is no general method of finding an integrating factor. But we shall discuss some rule of finding integrating factor.

Rules for finding the integrating factors :

I By inspection : Some times an integrating factor of given equation $M(x, y)\,dx + N(x, y)\,dy = 0$ can be found out by inspection. By rearranging the term of the given equation and dividing it by a suitable function of x and y, the equation will contain several parts which are integrable easily. For this purpose the following exact differentials should be remembered.

1. $d(x, y) = x\,dy + y\,dx$
2. $d[\log(x, y)] = \dfrac{x\,dy + y\,dx}{xy}$
3. $d\left(\dfrac{y}{x}\right) = \dfrac{x\,dy - y\,dx}{x^2}$
4. $d\left(\dfrac{x}{y}\right) = \dfrac{y\,dx - x\,dy}{y^2}$
5. $d\left[\log\left(\dfrac{x}{y}\right)\right] = \dfrac{y\,dx - x\,dy}{xy}$
6. $d\left[\log\left(\dfrac{y}{x}\right)\right] = \dfrac{x\,dy - y\,dx}{xy}$
7. $d\left(\dfrac{-1}{xy}\right) = \dfrac{x\,dy + y\,dx}{x^2y^2}$
8. $d\left(\dfrac{x^2}{y}\right) = \dfrac{2yx\,dx - x^2\,dy}{y^2}$
9. $d\left(\dfrac{x^2}{y^2}\right) = \dfrac{2y^2x\,dx - 2y\,x^2\,dy}{y^4}$
10. $d\left(\dfrac{e^x}{y}\right) = \dfrac{y\,e^x\,dx - e^x\,dy}{y^2}$
11. $d\left(\tan^{-1}\dfrac{y}{x}\right) = \dfrac{x\,dy - y\,dx}{x^2 + y^2}$
12. $d\left(\dfrac{1}{2}\log\dfrac{x+y}{x-y}\right) = \dfrac{x\,dy - y\,dx}{x^2 - y^2}$

II If the differential equation $M\,dx + N\,dy = 0$ is homogeneous and $Mx + Ny \neq 0$ then $\frac{1}{Mx+Ny}$ is an integrating factor. If $Mx + Ny = 0$ then $\frac{1}{xy}$ or $\frac{1}{x^2}$ or $\frac{1}{y^2}$ are the integrating factors.

Example 1 : Solve $\left(3xy^2 - y^3\right)dx - \left(2x^2y - xy^2\right)dy = 0$

Solution : The given differential equation is $\left(3xy^2 - y^3\right)dx - \left(2x^2y - xy^2\right)dy = 0$ ….(1)

Comparing (1) with $M\,dx + N\,dy = 0$, we have $M = 3xy^2 - y^3$ and $N = -2x^2y + xy^2$.

Then $\frac{\partial M}{\partial y} = 6xy - 3y^2$, $\frac{\partial N}{\partial x} = -4xy + y^2$. As $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Therefore equation (1) is not an exact equation. Since equation (1) is homogeneous in *x* and *y* and

$Mx + Ny = x^2y^2 \neq 0$

Therefore I.F. $= \frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$

Multiplying both sides of equation (1) by $\frac{1}{x^2y^2}$, we get $\left(\frac{3}{x} - \frac{y}{x^2}\right)dx - \left(\frac{2}{y} - \frac{1}{x}\right)dy = 0$ ……(2)

Now, equation (2) is exact.

Therefore the solution is $\int\limits_{y\,\text{constant}} \left(\frac{3}{x} - \frac{y}{x^2}\right)dx - \int \frac{2}{y}dy = c$

$\Rightarrow \quad 3\log x + \frac{y}{x} - 2\log y = c$

$\Rightarrow \quad \log\frac{x^3}{y^2} + \frac{y}{x} = c$ which is the required solution.

III If $M(x,y) = f(xy)\,y$ and $N(x,y) = g(xy)\,x$ for some functions *f* and *g*, then $\frac{1}{Mx-Ny}$ is an integrating factor providing $Mx - Ny \neq 0$.

In case $Mx - Ny = 0$, then $Mdx + Ndy = 0$ is already an exact equation.

Example 2 : Solve $\left(x^2y^3 + xy^2 + y\right)dx + \left(x^3y^2 - x^2y + x\right)dy = 0$

Solution : The given differential equation is $\left(x^2y^3 + xy^2 + y\right)dx + \left(x^3y^2 - x^2y + x\right)dy = 0$ ……(1)

Comparing (1) with $M\,dx + N\,dy = 0$, we get $M = x^2y^3 + xy^2 + y$, $N = x^3y^2 - x^2y + x$

Clearly $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Thus (1) is not exact.

Now equation (1) can be written as $\left(x^2y^2+xy+1\right)y\,dx+\left(x^2y^2-xy+1\right)x\,dy=0$

Now $Mx-Ny=2x^2y^2\neq 0$

$\therefore$ I.F. $=\dfrac{1}{Mx-Ny}=\dfrac{1}{2x^2y^2}$

Multiplying both sides of equation (1) by $\dfrac{1}{2x^2y^2}$, we get

$$\frac{1}{2}\left(y+\frac{1}{x}+\frac{1}{x^2y}\right)dx+\frac{1}{2}\left(x-\frac{1}{y}+\frac{1}{xy^2}\right)dy=0 \qquad \ldots\ldots(2)$$

Equation (2) is exact now.

$\therefore$ the solution is $\dfrac{1}{2}\displaystyle\int_{y\,\text{constant}}\left(y+\frac{1}{x}+\frac{1}{x^2y}\right)dx+\frac{1}{2}\int-\frac{1}{y}dy=c$

$\Rightarrow$ $\dfrac{1}{2}\left(xy+\log x-\dfrac{1}{xy}\right)-\dfrac{1}{2}\log y=c$

$\Rightarrow$ $xy+\log\dfrac{x}{y}-\dfrac{1}{xy}=c$ which is the required solution.

IV If the equation $M\,dx+N\,dy=0$ is such that $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}$ is a function of x only $=f(x)$ say then $e^{\int f(x)dx}$ is an integrating factor.

V If the equation $M\,dx+N\,dy=0$ is such that $\dfrac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M}$ is a function of y only $=f(y)$ (say) then $e^{\int f(y)dy}$ is an integrating factor.

VI If the equation $\dfrac{dy}{dx}=f(x,y)$ is of the form $x^ay^b\left[My\,dx+Nx\,dy\right]+x^r\,y^s\left[py\,dx+qx\,dy\right]=0,$ where a,b,M,N,r,s,p and q are all constants, then I.F. $=x^h\,y^k$, where h and k are chosen such that after multiplying the given differential equation by I.F. it becomes exact. This exact differential equation can be solved by the above described method.

Example 3. Solve $\left(y+\frac{1}{3}y^3+\frac{1}{2}x^2\right)dx+\frac{1}{4}\left(1+y^2\right)x\,dy=0$.

Solution : Here $M=y+\frac{1}{3}y^3+\frac{1}{2}x^2,\ \ N=\frac{1}{4}\left(1+y^2\right)x$

Clearly $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, so equation is not exact.

Now $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{4}{\left(1+y^2\right)x}\left[\left(1+y^2\right)-\frac{1}{4}\left(1+y^2\right)\right]$

$=\frac{4}{\left(1+y^2\right)x}\frac{3}{4}\left(1+y^2\right)=\frac{3}{x}$, which is a function of *x* only ……(1)

Therefore integrating factor $=e^{\int\frac{3}{x}dx}=e^{3\log x}=x^3$.

Now, multiplying the equation by integrating factor x^3, we get

$$\left(x^3y+\frac{1}{3}x^3y^3+\frac{1}{2}x^5\right)dx+\frac{1}{4}\left(1+y^2\right)x^4\,dy=0 \qquad \text{……(2)}$$

Now equation (2) is exact. Hence the solution is

$$\int\limits_{y(\text{const.})}\left(x^3y+\frac{1}{3}x^3y^3+\frac{x^5}{2}\right)dx=c \quad\Rightarrow\quad \frac{x^4}{4}y+\frac{1}{12}x^4y^3+\frac{x^6}{12}=c$$

$$\Rightarrow \quad 3x^4y+x^4y^3+x^6=c \qquad (\text{where } 12c=c)$$

Example 4. Solve $\left(xy^3+y\right)dx+2\left(x^2y^2+x+y^4\right)dy=0$

Solution : Here $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, so given equation is not exact.

Now $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{1}{y\left(xy^2+1\right)}\left\{4xy^2+2-3xy^2-1\right\}$

$=\frac{1}{y\left(xy^2+1\right)}\left(xy^2+1\right)=\frac{1}{y}$, which is a function of *y* alone ……(1)

Therefore integrating factor $=e^{\int\frac{1}{y}dy}=e^{\log y}=y$.

Now, multiplying the equation by integrating factor *y*, we obtain

$$y^2\left(xy^2+1\right)dx+2y\left(x^2y^2+x+y^4\right)dy=0 \qquad \text{……(2)}$$

This equation (2) is exact. Hence the solution is

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$$\int y^2\left(xy^2+1\right)dx+\int 2y^5\,dy=C$$

$\Rightarrow \quad y^4\dfrac{x^2}{2}+xy^2+2\dfrac{y^6}{6}=c$

$\Rightarrow \quad 3x^2y^4+6xy^2+2y^6=c \qquad (\text{where } 6c=c)$

Example 5. Solve $x(3ydx+2xdy)+8y^4(ydx+3xdy)=0$

Solution : We can write $\left(3xy+8y^5\right)dx+\left(2x^2+24xy^4\right)dy=0$ .....(1)

Here $\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$, so equation is not exact.

Let the integrating factor be x^hy^k. Therefore, multiplying by x^hy^k, we get

$\left(3x^{h+1}y^{k+1}+8y^{k+5}x^h\right)dx+\left(2x^{h+2}y^k+24x^{h+1}y^{k+4}\right)dy=0$ .....(2)

Now $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$ gives $3(k+1)x^{h+1}y^k+8(k+5)y^{k+4}x^h=2(h+2)x^{h+1}y^k+24(h+1)x^hy^{k+1}$

On equating coefficients of same powers, we obtain $3(k+1)=2(h+2)$ ......(3)

and $8(k+5)=24(h+1)$. ......(4)

On solving (3) and (4) for *h* and *k*, we get $h=1,\ k=1$.

Thus from (2), we get $\left(3x^2y^2+8xy^6\right)dx+\left(2x^3y+24x^2y^5\right)dy=0$

which is exact. Thus solution is $\int\left(3x^2y^2+8xy^6\right)dx=c \quad\Rightarrow\quad x^3y^2+4x^2y^6=c$

Exercise 2.1

Solve the following differential equations :

1. $\left(x^2-ay\right)dx=\left(ax-y^2\right)dy$
2. $y\sin 2x\,dx-\left(1+y^2+\cos^2 x\right)dy=0$
3. $\left(x^2+y^2\right)dx-2xy\,dy=0$
4. $\left(xy^2+2x^2y^3\right)dx+\left(x^2y-x^3y^2\right)dy=0$
5. $\left(x^2+y^2+2x\right)dx+2y\,dy=0$
6. $\left(3x^2y^4+2xy\right)dx+\left(2x^3y^3-x^2\right)dy=0$

Answers

1. $x^3 - 3axy + y^3 = c$
2. $3y\cos 2x + 6y + 2y^3 = c$
3. $x^2 - y^2 = cx$
4. $-\frac{1}{xy} + \log\frac{x^2}{y} = c$
5. $x^2e^x + y^2e^x = c$
6. $x^3y^3 + x^2 = c\,y$

Separable equations : An equation of the form $F(x)G(y)\,dx + f(x)\,g(y)\,dy = 0$ …..(*)

is called an equation with variables separable or simply a separable equation. e.g., the equation $(x-4)y^4\,dx - x^3(y^2-3)\,dy = 0$ is a separable equation.

In general the separable equation (*) is not exact, but it possesses an obvious integrating factor, namely $\frac{1}{f(x)G(y)}$. For if we multiply equation (*) by this expression, we separate the variables, reducing (*) to the essentially equivalent equation $\frac{F(x)}{f(x)}dx + \frac{g(y)}{G(y)}dy = 0$ ….(**)

This equation is exact, since $\frac{\partial}{\partial y}\left[\frac{F(x)}{f(x)}\right] = 0 = \frac{\partial}{\partial x}\left[\frac{g(y)}{G(y)}\right]$

Denoting $F(x)/f(x)$ by $M(x)$ and $g(y)/G(y)$ by $N(y),$ equation (**) takes the form $M(x)dx + N(y)dy = 0$. Since M is a function of x only and N is a function of y only, we see at once that a one-parameter family of solutions is $\int M(x)\,dx + \int N(y)\,dy = c$ …..(1)

where c is the arbitrary constant.

Since we obtained the separated exact equation (**) from the non exact equation (*) by multiplying (*) by the integrating factor $\frac{1}{f(x)G(y)},$ solutions may have been lost or gained in this process. We now consider this more carefully. In formally multiplying by the integrating factor $\frac{1}{f(x)G(y)},$ we actually divided by $f(x)G(y)$. We did this under the tacit assumption that neither $f(x)$ nor $G(y)$ is zero ; and, under this assumption, we proceeded to obtain the one-parameter family of solutions given by (1). Now, we should investigate the possible loss or gain of solutions that may have occurred in this formal process. In particular, regarding y as the dependent variable as usual, we consider the situation that occurs if $G(y)$ is zero. Writing the original differential equation (*) in the derivative form

$$f(x)\,g(y)\frac{dy}{dx} + F(x)\,G(y) = 0\,.$$

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We immediately note the following. If y_0 is any real number such that $G(y_0)=0,$ then $y=y_0$ is a (constant) solution of the original differential equation and this solution may (or may not) have been lost in the formal separation process.

In finding a one-parameter family of solutions a separable equation, we shall always make the assumption that any factor by which we divide in the formal separation process are not zero. Then we must find the solutions $y=y_0$ of the equation $G(y)=0$ and determine whether any of these are solutions of the original equation which were lost in the formal separation process.

Example : Solve the equation $(x-4)\,y^4\,dx-x^3(y^2-3)\,dy=0$

Solution : The equation is separable ; separating the variables by dividing by $x^3y^4,$ we obtain

$$\frac{(x-4)}{x^3}dx-\frac{(y^2-3)}{y^4}dy=0$$

or $$(x^{-2}-4x^{-3})\,dx-(y^{-2}-3y^{-4})\,dy=0$$

Integrating, we have the one-parameter family of solutions

$$-\frac{1}{x}+\frac{2}{x^2}+\frac{1}{y}-\frac{1}{y^3}=c$$

where *c* is the arbitrary constant. In dividing by x^3y^4 in the separation process, we assumed that $x^3\neq 0$ and $y^4\neq 0$. We now consider the solution $y=0$ of $y^4=0$. It is not a member of the one-parameter family of solutions which we obtained. However, writing the original differential equation of the problem in the derivative form $\frac{dy}{dx}=\frac{(x-4)\,y^4}{x^3(y^2-3)}$

it is obvious that $y=0$ is a solution of the original equation. We conclude that it is a solution which was lost in the separation process.

Homogeneous equation : The first order differential equation $M(x,y)\,dx+N(x,y)\,dy=0$ is said to homogeneous if, when written in the derivative form $\frac{dy}{dx}=f(x,y),$ there exists a function *g*, such that $f(x,y)$ can be expressed in the form $g\left(\frac{y}{x}\right)$. e.g., the equation $\left(y+\sqrt{x^2+y^2}\right)dx-xdy=0$ ….(1)

we written in the form $\frac{dy}{dx}=\frac{y+\sqrt{x^2+y^2}}{x}\quad=\frac{y}{x}+\sqrt{1+\left(\frac{y}{x}\right)^2}$

This is obviously of the form $g\left(\frac{y}{x}\right)$

$\therefore$ (1) is homogeneous equation.

Remark : If M and N in $M(x,y)\,dx + N(x,y)\,dy = 0$ are both homogeneous functions of the same degree n, then the differential equation is a homogeneous differential equation i.e., multiply each variable by a parameter λ, we find $M(\lambda x, \lambda y) = \lambda^n M(x,y)$ and $N(\lambda x, \lambda y) = \lambda^n N(x,y)$.

Thus $\frac{M(\lambda x,\lambda y)}{N(\lambda x,\lambda y)} = \frac{M(x,y)}{N(x,y)}$. Therefore the differential equation $\left(y+\sqrt{x^2+y^2}\right)dx - x\,dy = 0$

$$M(x,y) = y + \sqrt{x^2+y^2},\ N = -x$$

$$N(\lambda x,\lambda y) = -\lambda x = \lambda N(x,y)$$

$$M(\lambda x,\lambda y) = \lambda y + \sqrt{(\lambda x)^2 + (\lambda y)^2} = \lambda\left(y+\sqrt{x^2+y^2}\right) = \lambda M(x,y)$$

Clearly, N is homogeneous of degree 1. We see that M is also homogeneous of degree 1. Thus we conclude that the equation $\left(y+\sqrt{x^2+y^2}\right)dx - xdy = 0$ is homogeneous.

Theorem : If $M(x,y)\,dx + N(x,y)\,dy = 0$ …..(1)

Is a homogeneous equation, then the change of variables $y = vx$ transforms (1) into a separable equation in the variables v and x.

Homogeneous function : A function F is called homogeneous of degree n if $F(tx,ty) = t^n F(x,y)$.

Differential equation reducible into homogeneous : Consider the equation

$$\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1} \quad \text{…..(I)}$$

where $\frac{a}{b} \neq \frac{a_1}{b_1}$. If c and c_1 are both zero, the equation is homogeneous. If c and c_1 are not both zero, then we change the variables so that constant terms are no longer present, by the substitution

$$x = X + h, \qquad y = Y + k \quad \text{…..(II)}$$

and (I) reduces to $$\frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{a_1(X+h)+b_1(Y+k)+c_1}$$

$$\Rightarrow \quad \frac{dY}{dX} = \frac{aX+bY+(ah+bk+c)}{a_1X+b_1Y+(a_1h+b_1k+c_1)} \quad \text{…..(III)}$$

Now choose h and k such that $ah+bk+c=0$ and $a_1h+b_1k+c_1=0$ …..(IV)

Then (III) reduces to $\frac{dY}{dX} = \frac{aX+bY}{a_1X+b_1Y}$, which is homogeneous and can be solved by substitution $Y = vx$. Replacing X and Y in the solution so obtained by $x-h,\quad y-k$ from (II) respectively we will

get the required solution in terms of x and y the original variables.

Special Case : When $\frac{a}{a_1} = \frac{b}{b_1} = k$ in this case we can not solve the equation given by (IV) above and differential equation is of the form $\frac{dy}{dx} = \frac{ax+by+c}{kax+kby+c_1}$.

In this case, the differential equation is solve by putting $v = ax + by$.

Example : Solve $\frac{dy}{dx} = \frac{(x+y-2)}{y-x-4}$.

Solution : Let $x = X + h$ and $y = Y + k$ so that $\frac{dy}{dx} = \frac{dY}{dX}$ ....(1)

Then given equation becomes $\frac{dY}{dX} = \frac{X+Y+(h+k-2)}{Y-X+(k-h-4)}$ ....(2)

Choose h, k such that $h + k - 2 = 0$ and $k - h - 4 = 0$ .....(3)

Solving (3) ; $h = -1, k = 3$. Then (1) gives $X = x + 1$ and $Y = y - 3$ .....(4)

Using (3), (2) becomes $\frac{dY}{dX} = \frac{X+Y}{Y-X}$ .....(5)

Let $Y - vX$ so that $\frac{dY}{dX} = v + X\frac{dv}{dX}$ ....(6)

From (5) and (6), we have $v + X\frac{dv}{dX} = \frac{1+v}{v-1}$

$\Rightarrow \quad X\frac{dv}{dX} = \frac{1+2v-v^2}{v-1}$

$\Rightarrow \quad \frac{v-1}{1+2v-v^2} \cdot dv = \frac{dX}{X}$

On solving, we get $\quad \log(1+2v-v^2) = c$

$\Rightarrow \quad X^2(1+2v-v^2) = c$

$\Rightarrow \quad X^2\left(1+\frac{2Y}{X}-\frac{Y^2}{X^2}\right) = c$

$\Rightarrow \quad X^2 + 2XY - Y^2 = c$

$\Rightarrow \quad (x+1)^2 + 2(x+1)(y-3) - (y-3)^2 = c$ where c being arbitrary constant.

Exercise 2.2

Solve each of the following differential equations :

1. $4xy\,dx+(x^2+1)\,dy=0$
2. $\tan\theta\,dr+2r d\theta=0$
3. $(2x-5y)\,dx+(4x-y)\,dy=0,\;\; y(1)=4$
4. $(x^2-3y^2)\,dx+2xy\,dy=0$
5. $\left(6x+2y-10\right)dy-\left(2x+9y-20\right)dx=0$
6. $\left(4x+6y+5\right)dy-\left(3y+2x+4\right)dx=0$

Answers

1. $(x^2+1)\,y=c$
2. $r\sin^2\theta=c$
3. $(2x+y)^2=12(y-x)$
4. $y^2-x^2=cx^3$
5. $x+2y-5=c\left(2x-y\right)^2$
6. $\dfrac{2}{7}\left(2x+3y\right)-\dfrac{9}{49}\log\left(14x+21y+22\right)=x+c$

Linear equation : A first order ordinary differential equation is linear in the dependent variable y and independent variable x if it is, or can be written in the form $\dfrac{dy}{dx}+P(x)y=Q(x)$ …..(*)

e.g., the equation $x\dfrac{dy}{dx}+(x+1)y=x^3$

is a first order linear differential equation, for it can be written as $\dfrac{dy}{dx}+\left(1+\dfrac{1}{x}\right)y=x^2$

which is of the form (*).

Theorem : The linear differential equation $\dfrac{dy}{dx}+P(x)\,y=Q(x)$ ……(*)

has an integrating factor of the form $e^{\int P(x)\,dx}$.

A one-parameter family of solutions of this equation is $y\cdot e^{\int P\,dx}=\int e^{\int P(x)\,dx}\,Q(x)\,dx+c$.

Furthermore, this one-parameter family of solutions of the linear equation (*) includes all solutions of (*).

Example : Solve $\dfrac{dy}{dx}+\left(\dfrac{2x+1}{x}\right)y=e^{-2x}$.

Here $P(x)=\dfrac{2x+1}{x}$, $Q(x)=e^{-2x}$.

And hence I.F. $=e^{\int P(x)\,dx}=e^{\int\left(\frac{2x+1}{x}\right)dx}=e^{2x+\ln x}=xe^{2x}$

Solution is $y \cdot \text{I.F.} = \int \text{I.F.}\, Q(x)\, dx + c$

$\Rightarrow \quad y\, x e^{2x} = \int x e^{2x} \cdot e^{-2x} + c$

$\Rightarrow \quad xy\, e^{2x} = \dfrac{x^2}{2} + c$

$\Rightarrow \quad y = \dfrac{1}{2} x e^{-2x} + \dfrac{c}{x} e^{-2x}$

where c is an arbitrary constant.

Bernoulli equations : An equation of the form $\dfrac{dy}{dx} + P(x)y - Q(x)y^n$, $\quad n \neq 0,1$ is called a Bernoulli equation. Where as if $n = 0$ or 1, then Bernoulli equation is actually a linear equation.

Theorem : Suppose $n \neq 0$, or 1. Then the transformation $v = y^{1-n}$ reduces the Bernoulli equation $\dfrac{dy}{dx} + P(x)y = Q(x)y^n$ to a linear equation in v.

Example : Solve $\dfrac{dy}{dx} + y = xy^3$ …..(1)

Solution : Let $v = y^{1-3} = y^{-2}$, then $\dfrac{dv}{dx} = -2y^{-3} . \dfrac{dy}{dx}$ …..(2)

We multiply the equation (1) by y^{-3}, we get $y^{-3} \dfrac{dy}{dx} + y^{-2} = x$ …..(3)

By (2), (3) becomes $-\dfrac{1}{2} \dfrac{dv}{dx} + v = x$

$\Rightarrow \quad \dfrac{dv}{dx} - 2v = -2x$ …..(4)

I.F. $= e^{\int P(x)\, dx} = e^{\int -2\, dx} = e^{-2x}$

$\therefore$ solution of (4) is

$v \cdot e^{-2x} = \int e^{-2x} \cdot (-2x)\, dx + c$

$= -2\int x e^{-2x}\, dx + c$

$$\Rightarrow \quad v\cdot e^{-2x} = -2\left[x\frac{e^{-2x}}{-2} - \frac{e^{-2x}}{(-2)^2}\right] + c$$

$$\Rightarrow \quad v\cdot e^{-2x} = xe^{-2x} + \frac{e^{-2x}}{2} + c$$

$$\Rightarrow \quad v = x + \frac{1}{2} + c\,e^{2x}$$

where c is an arbitrary constant. But $v = y^{-2} = \frac{1}{y^2}$

$$\therefore \quad \frac{1}{y^2} = x + \frac{1}{2} + c\,e^{2x}$$

Exercise 2.3

Solve the following differential equations :

1. $\frac{dy}{dx} + \frac{3y}{x} = 6x^2$
2. $\frac{dy}{dx} - \frac{y}{x} = \frac{-y^2}{x}$

Answers

1. $y = x^3 + cx^{-3}$
2. $y = (1 + cx^{-1})^{-1}$

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Assignment-2

---------------------- S C Q ---------------------

1. The equation

$(\alpha xy^3 + y\cos x)dx + (x^2y^2 + \beta\sin x)dy = 0$

is exact for

1. $\alpha = \frac{3}{2}$, $\beta = 1$ 2. $\alpha = 1$, $\beta = \frac{3}{2}$
3. $\alpha = \frac{2}{3}$, $\beta = 1$ 4. $\alpha = 1$, $\beta = \frac{2}{3}$

2. An integrating factor for

$\cos y \sin 2x dx + (\cos^2 y - \cos^2 x)\, dx = 0$

1. $\sec^2 y + \sec y \tan y$
2. $\tan^2 y + \sec y \tan y$
3. $1/(\sec^2 y + \sec y \tan y)$
4. $1/(\tan^2 y + \sec y \tan y)$

3. Which of the following is not an integrating factor of $xdy - ydx = 0$

1. $\frac{1}{x^2}$ 2. $\frac{1}{x^2+y^2}$
3. $\frac{1}{xy}$ 4. $\frac{x}{y}$

4. The general solution of the differential equation $\frac{dy}{dx} + \tan y \tan x = \cos x \sec y$ is

1. $2\sin y = (x + c - \sin x \cos x)\sec x$
2. $\sin y = (x + c)\cos x$
3. $\cos y = (x + c)\sin x$
4. $\sec y = (x + c)\cos x$

5. The differential equation

$\frac{dy}{dx} = h(a-y)(b-y)$, when solved with the condition $y(0) = 0$, yields the result

1. $\frac{b(a-y)}{a(b-y)} = e^{(a-b)hx}$
2. $\frac{b(a-x)}{a(b-x)} = e^{(b-a)hy}$
3. $\frac{a(b-y)}{b(a-y)} = e^{(a-b)hx}$
4. $x, y = he$

6. The differential equation

$(3a^2x^2 + by\cos x)\, dx + (2\sin x - 4ay^3)dy = 0$
is exact for

1. $a = 3,\ b = 2$ 2. $a = 2,\ b = 3$
3. $a = 3,\ b = 4$ 4. $a = 2,\ b = 5$

7. For the differential equation $xy' - y = 0$, which of the following function is not an integrating factor ?

1. $\frac{1}{x^2}$ 2. $\frac{1}{y^2}$
3. $\frac{1}{xy}$ 4. $\frac{1}{x+y}$

8. For $a, b, c \in \mathbb{R}$, if the differential equation

$(ax^2 + bxy + y^2)dx + (2x^2 + cxy + y^2)dy = 0$

is exact , then

1. $b = a$, $c = 20$ 2. $b = 4$, $c = 2$
3. $b = 2$, $c = 4$ 4. $b = 2$, $a = 2c$

9. The differential equation

$(1 + x^2y^3 + \alpha x^2y^2)dx + (2 + x^3y^2 + x^3y)dy = 0$
is exact if α equals

1. $\frac{1}{2}$
2. $\frac{3}{2}$
3. 2
4. 3

10. An integrating factor for the differential equation $(2xy+3x^2y+6y^3)dx+(x^2+6y^2)dy=0$ is
    1. x^3
    2. y^3
    3. e^{3x}
    4. e^{3y}

11. If y^a is an integrating factor of the differential equation $2xydx-(2x^2-y^2)dy=0$ then the value of a is
    1. –4
    2. 4
    3. –3
    4. 3

12. Consider the differential equation $(x+y+1)dx+(2x+2y+1)dy=0$. Which of the following statement is true ?
    1. The differential equation is linear.
    2. The differential equation is exact.
    3. e^{x+y} is an integrating factor of the differential equation.
    4. A suitable substitution transforms the differential equation to the variables separable form.

13. Consider the differential equation $2\cos(y^2)dx-xy\sin(y^2)dy=0$. Then
    1. e^x is an integrating factor.
    2. e^{-x} is an integrating factor.
    3. $3x$ is an integrating factor.
    4. x^3 is an integrating factor.

14. One of the integrating factor of the differential equation $(y^2-3xy)dx+(x^2-xy)dy=0$ is
    1. $\frac{1}{x^2y^2}$
    2. $\frac{-1}{2x^2y}$
    3. $\frac{1}{xy^2}$
    4. $\frac{1}{xy}$

15. An integrating factor of $x\frac{dy}{dx}+(3x+1)y=xe^{-2x}$ is
    1. xe^{3x}
    2. $3xe^x$
    3. xe^x
    4. x^3e^x

16. The solution of differential equation $\frac{dy}{dx}=\frac{x+y+1}{x+y-1}$, $y\left(\frac{2}{3}\right)=\frac{1}{3}$ is
    1. $y=x+\log(x+y)-\frac{1}{3}$
    2. $y=x-\log(x+y)+\frac{2}{3}$
    3. $y=x+\log(x+y)+\frac{2}{3}$
    4. $y=x-\log(x+y)+1$

---------------------- **M C Q** --------------------

1. For the differential equation $xy'-y=0$, which of the following function is an integrating factor
    1. $\frac{1}{x^2}$
    2. $\frac{1}{y^2}$
    3. $\frac{1}{xy}$
    4. $\frac{x}{y}$

2. Consider the differential equation $\frac{dy}{dx}+P(x)y=Q(x)$ is
    1. a linear differential equation

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2. a non linear differential equation
3. $e^{\int P(x)dx}$ is an I.F.
4. $e^{\int Q(x)dx}$ is an I.F.

3. Consider the differential equation

$\frac{dy}{dx}+P(x)y=Q(x)y^n$ is

1. a linear differential equation for $n=0$
2. a linear differential equation for $n=1$
3. a Bernoulli equation for $n\neq 0,1$
4. a Bernoulli equation for $n=10$

Answers

----------------------- **S C Q** --------------------

1. 3	2. 1	3. 4	4. 2
5. 1	6. 1	7. 4	8. 2
9. 2	10. 3	11. 3	12. 4
13. 4	14. 2	15. 1	16. 1

---------------------- **M C Q** --------------------

1. 1,2,3	2. 1,3	3. 1,2,3,4

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Chapter - 3

Basic theory of linear differential equations

Def. Linear Differential Equation : A differential equation, in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together, is called linear differential equation.

Def. Linear Differential Equations with constant coefficients : A linear differential equation, in which the coefficients of all the derivatives and dependent variable are constant is called a linear differential equation with constant coefficients.

Thus, the general linear differential equation of *n*th order is of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + a_{n-1} \frac{dy}{dx} + a_n y = X \qquad ......(1)$$

where $a_1, a_2, ..., a_{n-1}, a_n$ are constants and X is a function of x only.

Note : In this chapter, we shall deal with linear differential equations with constant coefficients.

Differential operator *D* :

Let us denote $\frac{d}{dx}$ by D, $\frac{d^2}{dx^2}$ by D^2 and so on.

The symbols D, D^2 ,............., are called differential operators and for any function y, we have

$$Dy = \frac{dy}{dx} \ , \quad D^2 y = \frac{d^2 y}{dx^2} \text{ and so on.}$$

For example, $D(x^2) = 2x$, $D^2(x^2) = 2$, $D^4(e^{2x}) = 16 e^{2x}$

Symbolic Form of Linear Differential Equation :

If we use the notations $\frac{d}{dx} \equiv D, \frac{d^2}{dx^2} \equiv D^2, ..., \frac{d^n}{dx^n} \equiv D^n$, then (1) becomes

$$D^n y + a_1 D^{n-1} y + ... + a_n y = X$$

or $$(D^n + a_1 D^{n-1} + ... + a_n) y = X$$

or $$f(D) y = X \qquad ......(2)$$

where $$f(D) = D^n + a_1 D^{n-1} + ... + a_n$$

The form of the Linear Differential Equation (1), in terms of the differential operator D as shown in (2) is called symbolic form of (1).

Theorem : If $y_1, y_2, ..., y_n$ are solutions of a linear differential equation $(D^n + a_1 D^{n-1} + ... + a_n) y = 0$, then $y = c_1 y_1 + c_2 y_2 + ... + c_n y_n$ is also a solution of the linear differential equation where $c_1, c_2, ..., c_n$ are arbitrary constant.

Complete Solution of a Linear Differential Equation :

Let the given linear differential equation of *n*th order be

$$D^n y + a_1 D^{n-1} y + ... + a_n y = X \qquad ......(1)$$

Suppose $y_1, y_2, ..., y_n$ are *n* independent solutions of the differential equation

$$D^n y + a_1 D^{n-1} y + ... + a_n y = 0 \qquad ......(2)$$

Then as shown in the previous theorem that $y = c_1 y_1 + c_2 y_2 + ... + c_n y_n$ is also a solution of (2) i.e.,

$$D^n (c_1 y_1 + c_2 y_2 + ... + c_n y_n) + a_1 D^{n-1} (c_1 y_1 + c_2 y_2 + ... + c_n y_n) + ...$$
$$+ a_n (c_1 y_1 + c_2 y_2 + ... + c_n y_n) = 0 \qquad ......(3)$$

Now, Let $y = v$ be a particular solution of (1) so that $D^n v + a_1 D^{n-1} v + ... + a_n v = X$ ......(4)

Adding (3) and (4), we have

$$D^n (c_1 y_1 + c_2 y_2 + ... + c_n y_n + v) + a_1 D^{n-1} (c_1 y_1 + c_2 y_2 + ... + c_n y_n + v) + ...$$
$$.... + a_n (c_1 y_1 + c_2 y_2 + ... + c_n y_n + v) = X \qquad ......(5)$$

This shows that $c_1 y_1 + c_2 y_2 + ... + c_n y_n + v$ is a solution of (1). Since it involves *n* arbitrary constants (equal to the order of linear differential equation), so it is a complete solution or general solution of (1) The part $c_1 y_1 + c_2 y_2 + ... + c_n y_n$ is known as complementary function (*C.F*) and *v* is called particular integral (*P.I*).

Thus complete solution (*C.S.*) of (1) is given by $C.S. = C.F. + P.I.$

Let us define the terms complementary function and particular integral in a formal way.

Def. Complementary Function : Let the given linear differential equation of *n*th order be

$$D^n y + a_1 D^{n-1} y + ... + a_n y = X \qquad ......(1)$$

The general solution of equation, obtained from (1) by taking $X = 0$ i.e. $D^n y + a_1 D^{n-1} y + ... + a_n y = 0$ is called Complementary Function of (1).

Def. Particular Integral : Let the given linear differential equation of *n*th order be

$$D^n y + a_1 D^{n-1} y + ... + a_n y = X \qquad ......(1)$$

Any solution of (1) involving no arbitrary constant is called a Particular Integral of (1).

Remark : In the general solution (or complete solution) $y = c_1 y_1 + c_2 y_2 + ... + c_n y_n + v$, the Particular Integral i.e. *v* is due to the R.H.S. *X* in linear differential equation (1).

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Therefore if $X = 0$ in (1), then its complete solution will not the contain the Particular Integral v and therefore complete solution is given by $y = c_1 y_1 + c_2 y_2 + ... + c_n y_n$

i.e. Complete solution = Complementary Function

Remark : We conclude the above rules in the following table :

Sr. No.		Nature of roots of A.E.	Corresponding Part of C.F.
1.	(i)	One real root m_1	$c_1 e^{m_1 x}$
	(ii)	Two real and different roots m_1, m_2	$c_1 e^{m_1 x} + c_2 e^{m_2 x}$
	(iii)	Three real and different roots m_1, m_2, m_3	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$
2.	(i)	Two real and equal roots m_1, m_1	$(c_1 + c_2 x) e^{m_1 x}$
	(ii)	Three real and equal roots m_1, m_1, m_1	$(c_1 + c_2 x + c_3 x^2) e^{m_1 x}$
3.	(i)	One pair of complex roots $\alpha \pm i\beta$	$e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$
	(ii)	Two pairs of complex and equal roots $\alpha \pm i\beta, \alpha \pm i\beta$	$e^{\alpha x}[(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x]$
4.	(i)	One pair of surd roots $\alpha \pm \sqrt{\beta}$	$e^{\alpha x}(c_1 \cosh x\sqrt{\beta} + c_2 \sinh x\sqrt{\beta})$
	(ii)	Two pairs of surd and equal roots $\alpha \pm \sqrt{\beta}, \alpha \pm \sqrt{\beta}$	$e^{\alpha x}[(c_1 + c_2 x) \cosh x\sqrt{\beta} + (c_3 + c_4 x) \sinh x\sqrt{\beta}]$

Remarks : (i) When Auxiliary equation has one pair of complex roots $\alpha \pm i\beta$, then the corresponding part of complementary function is some times given as $c_1 e^{\alpha x} \cos(\beta x + c_2)$ or $c_1 e^{\alpha x} \sin(\beta x + c_2)$.

(ii) When Auxiliary equation has one pair of surd roots $\alpha \pm \sqrt{\beta}$, then the corresponding part of complementary function is sometimes given as $c_1 e^{\alpha x} \cosh(x\sqrt{\beta} + c_2)$ or $c_1 e^{\alpha x} \sinh(x\sqrt{\beta} + c_2)$.

Example 1 : Solve $\frac{d^3 y}{dx^3} - 4\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} + 18y = 0$

Solution : The given differential equation is

$$\frac{d^3 y}{dx^3} - 4\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} + 18y = 0 \qquad ......(1)$$

Equation (1) can be written as $(D^3 - 4D^2 - 3D + 18) y = 0$

$\therefore$ Auxiliary equation is $m^3 - 4m^2 - 3m + 18 = 0$

or $(m+2)(m-3)^2 = 0$

$\therefore$ $m = -2, 3, 3$

Thus we have three real roots out of which two are equal.

$\therefore$ The complete solution is $y = c_1 e^{-2x} + (c_2 + c_3 x)e^{3x}$

Exercise 3.1

1. $(D^4 + 6D^3 + 5D^2 - 24D - 36)y = 0$
2. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = 0$
3. $\dfrac{d^3y}{dx^3} - 13\dfrac{dy}{dx} - 12y = 0$
4. $\dfrac{d^4y}{dx^4} + 4y = 0$

Answers

1. $y = c_1 e^{2x} + c_2 e^{-2x} + (c_3 + c_4 x)e^{-3x}$
2. $y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x}$
3. $y = c_1 e^{-x} + c_2 e^{-3x} + c_3 e^{4x}$
4. $y = e^{-x}(c_1 \cos x + c_2 \sin x) + e^{x}(c_3 \cos x + c_4 \sin x)$

Particular Integrals

I Particular Integral (*P.I.*) of *f*(*D*)*y* = *X*

The given differential equation is

$$f(D)y = X \qquad \text{......(1)}$$

Let $y = v$ be a *P.I.* of (1). Then

$$f(D)v = X \qquad \text{......(2)}$$

Applying the operator $\dfrac{1}{f(D)}$ to both sides of (2), we get $\dfrac{1}{f(D)} f(D)v = \dfrac{1}{f(D)} X$

$$\Rightarrow \qquad v = \frac{1}{f(D)} X$$

Thus $\qquad P.I. = \dfrac{1}{f(D)} X$

Note : The expression for *P.I.* $\dfrac{1}{f(D)} X$ should not be written as $\dfrac{X}{f(D)}$ or $X\dfrac{1}{f(D)}$, as $\dfrac{1}{f(D)}$ is an operator, so it should be put on the left of the function.

II Prove that $\frac{1}{D-\alpha}X = e^{\alpha x}\int (e^{-\alpha x}X)dx$, **no arbitrary constant being added.**

III To determine the particular integral of *f*(*D*)*y* = *X* where

$F(D) = (D-\alpha_1)(D-\alpha_2).......(D-\alpha_n)$ and $\alpha_1, \alpha_2, ..., \alpha_n$ are constants.

$$P.I. = \frac{1}{f(D)}X \qquad = \frac{1}{(D-\alpha_1)(D-\alpha_2).......(D-\alpha_n)}(X)$$

$$= \left(\frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + ... + \frac{A_n}{D-\alpha_n}\right)X$$

[$A_1, A_2, ..., A_n$ are constants of partial fractions]

$$= A_1\frac{1}{D-\alpha_1}(X) + A_2\frac{1}{D-\alpha_2}(X) + ... + A_n\frac{1}{D-\alpha_n}(X)$$

$$= A_1e^{\alpha_1 x}\int Xe^{-\alpha_1 x}\,dx + A_2e^{\alpha_2 x}\int Xe^{-\alpha_2 x}\,dx + ... + A_ne^{\alpha_n x}\int Xe^{-\alpha_n x}\,dx \qquad \text{[By 2.3]}$$

IV Particular integral in some special cases :

(i) If $X = e^{ax}$, then $\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$ provided $f(a) \neq 0$

(ii) Case of failure : If $f(a) = 0$, then $\frac{1}{f(D)}e^{ax} = x\frac{1}{f'(a)}e^{ax}$ provided $f'(a) \neq 0$

Example 1 : Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{4x}$.

Solution : The given differential equation is $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{4x}$ ......(1)

Equation (1) can be written as $(D^2 - 5D + 6)y = e^{4x}$

$\therefore$ The auxiliary equation is $m^2 - 5m + 6 = 0 \quad \Rightarrow \quad m = 2, 3$

$\therefore$ $C.F. = c_1e^{2x} + c_2e^{3x}$

Now $P.I. = \frac{1}{D^2 - 5D + 6}e^{4x}$

$$= \frac{1}{16-20+6}e^{4x} = \frac{1}{2}e^{4x} \qquad \text{[Putting } D = 4\text{]}$$

$\therefore$ The complete solution of equation (1) is

$$y = C.F. + P.I$$

$$= c_1 e^{2x} + c_2 e^{3x} + \frac{e^{4x}}{2}$$

Example 2 : Solve $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$.

Solution : The given differential equation is

$$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x \qquad \text{......(1)}$$

Equation (1) can be written as

$$(D^2 + 1)y = 0$$

$\therefore$ Auxiliary equation is $\quad m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i$

$\therefore \quad C.F. = c_1 \cos x + c_2 \sin x$

Now, $\quad P.I. = \frac{1}{D^2+1} \operatorname{cosec} x$

$$= \frac{1}{(D-i)(D+i)} \operatorname{cosec} x$$

$$= \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right] \operatorname{cosec} x \qquad \text{[Making partial fractions]}$$

$$= \frac{1}{2i}\left[\frac{1}{D-i} \operatorname{cosec} x - \frac{1}{D+i} \operatorname{cosec} x\right]$$

$$= \frac{1}{2i}\left[e^{ix}\int e^{-ix} \operatorname{cosec} x \, dx - e^{-ix}\int e^{ix} \operatorname{cosec} x \, dx\right]$$

$$\left[\because \quad \frac{1}{D-\alpha} X = e^{\alpha x}\int e^{-\alpha x} X \, dx\right]$$

$$= \frac{1}{2i}\left[e^{ix}\int \frac{\cos x - i \sin x}{\sin x} dx - e^{-ix}\int \frac{\cos x + i \sin x}{\sin x} dx\right]$$

$$= \frac{1}{2i}\left[e^{ix}\int (\cot x - i)\, dx - e^{-ix}\int (\cot x + i)\, dx\right]$$

$$= \frac{1}{2i}\left[(e^{ix} - e^{-ix})\int \cot x \, dx - (e^{ix} + e^{-ix}) i \int dx\right]$$

$$= \left(\frac{e^{ix} - e^{-ix}}{2i}\right)\int \cot x \, dx - \left(\frac{e^{ix} + e^{-ix}}{2}\right)\int dx$$

$$= \sin x \log \sin x - \cos x \, . \, x$$

$\therefore$ The complete solution of equation (1) is

$$y = C.F. + P.I.$$

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i.e. $\quad y = c_1 \cos x + c_2 \sin x + \sin x \log \sin x - x\cos x$

V (i) If $X = \sin(ax+b)$, then $\dfrac{1}{f(D^2)}[\sin(ax+b)] = \dfrac{\sin(ax+b)}{f(-a^2)}$; provided $f(-a^2) \neq 0$

(ii) Case of failure : If $f(-a^2) = 0$, then $\dfrac{1}{f(D^2)}[\sin(ax+b)] = \dfrac{x}{f'(-a^2)}\sin(ax+b)$;

provided $f'(-a^2) \neq 0$.

VI (i) If $X = \cos(ax+b)$, then $\dfrac{1}{f(D^2)}[\cos(ax+b)] = \dfrac{\cos(ax+b)}{f(-a^2)}$; provided $f(-a^2) \neq 0$

(ii) Case of failure : If $f(-a^2) = 0$, then $\dfrac{1}{f(D^2)}[\cos(ax+b)] = \dfrac{x}{f'(-a^2)}\cos(ax+b)$;

provided $f'(-a^2) \neq 0$.

Working rule : To find the Particular integral of the type $\dfrac{1}{f(D)}\sin(ax+b)$ or $\dfrac{1}{f(D)}\cos(ax+b)$

Replace $\quad D^2$ by $-a^2$

$D^3 = D^2 . D$ by $-a^2 D$

$D^4 = D^2 . D^2$ by $(-a^2)^2$ and so on.

To simplify $\dfrac{1}{D+\alpha}\sin(ax+b)$ **proceed as follows :**

$$\frac{1}{D+\alpha}\sin(ax+b) = \frac{D-\alpha}{D^2-\alpha^2}\sin(ax+b) = \frac{D-\alpha}{-a^2-\alpha^2}\sin(ax+b)$$

$$= \frac{-1}{a^2+\alpha^2}\left(D\sin(ax+b) - \alpha\sin(ax+b)\right)$$

$$= \frac{-1}{a^2+\alpha^2}\left(\frac{d}{dx}\sin(ax+b) - \alpha\sin(ax+b)\right)$$

$$= \frac{-1}{a^2+\alpha^2}\left(a\cos(ax+b) - \alpha\sin(ax+b)\right)$$

Similarly proceed for $\dfrac{1}{D+\alpha}\cos(ax+b)$

Warning : Don't put anything in place of D.

Example 3 : Solve $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}-\frac{dy}{dx}-y=\cos 2x$

Solution : The given differential equation is $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}-\frac{dy}{dx}-y=\cos 2x$ ......(1)

Equation (1) can be written as $(D^3+D^2-D-1)y=\cos 2x$

$\therefore$ The auxiliary equation is $m^3+m^2-m-1=0$

or $m^2(m+1)-(m+1)=0$

or $(m^2-1)(m+1)=0$

or $(m-1)(m+1)(m+1)=0 \quad \Rightarrow \quad m=-1,-1,1.$

$\therefore \quad C.F. = c_1e^x+(c_2+c_3x)e^{-x}$

Now, $P.I.= \frac{1}{D^3+D^2-D-1}\cos 2x$

$=\frac{1}{-4D-4-D-1}\cos 2x$ [Putting $D^2=-2^2=-4$]

$= -\frac{1}{5}\frac{1}{D+1}\cos 2x$

$=-\frac{1}{5}(D-1)\frac{1}{D^2-1}\cos 2x$

$=-\frac{1}{5}(D-1)\frac{1}{-4-1}\cos 2x$ [Putting $D^2=-2^2=-4$]

$=\frac{1}{25}(D-1)\cos 2x \quad =\frac{1}{25}\left[D\cos 2x-\cos 2x\right]$

$=\frac{1}{25}\left[-2\sin 2x-\cos 2x\right]= -\frac{1}{25}\left[2\sin 2x+\cos 2x\right]$

$\therefore$ The complete solution of equation (1) is

$y = C.F. + P.I.$

or $y = c_1e^x+(c_2+c_3x)e^{-x}-\frac{1}{25}\left[2\sin 2x+\cos 2x\right]$

VII If $X = x^m$, then $\frac{1}{f(D)}x^m$ can be determined using following steps :

(i) Take lowest degree term common from $f(D)$ in denominator so that the remaining factor is of the form $\left[1\pm g(D)\right]$

(ii) Take $\left[1\pm g(D)\right]$ in numerator with negative power.

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(iii) Expand $\left[1 \pm g(D)\right]$ with negative power using Binomial Theorem up to the terms of D^m and neglect higher powers of D because their contribution in $P.I.$ will be zero.

(iv) Now operate with each term of the operator on x^m and simplify.

Note : The following binomial expansions are useful for expanding $[f(D)]^{-1}$ in ascending powers of D.

$$(1-D)^{-1} = 1 + D + D^2 + D^3 + \ldots\ldots\ldots\ldots$$

$$(1+D)^{-1} = 1 - D + D^2 - D^3 + \ldots\ldots\ldots$$

In general $$(1+D)^n = 1 + nD + \frac{n(n-1)}{2!}D^2 + \ldots\ldots\ldots\ldots$$

Example 4 : Solve $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} = x^2$

Solution : The given differential equation is

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 6\frac{dy}{dx} = x^2 \qquad \ldots\ldots(1)$$

The equation (1) can be written as $(D^3 - D^2 - 6D)y = x^2$

$\therefore$ Auxiliary equation is $m^3 - m^2 - 6m = 0$

or $m(m^2 - m - 6) = 0$

or $m(m-3)(m+2) = 0 \quad \Rightarrow \quad m = 0, -2, 3$

$\therefore$ $C.F. = c_1 e^{0x} + c_2 e^{-2x} + c_3 e^{3x} = c_1 + c_2 e^{-2x} + c_3 e^{3x}$

Now, $$P.I. = \frac{1}{D^3 - D^2 - 6D}x^2$$

$$= \frac{-1}{6D}\frac{1}{\left[1+\left(\frac{D}{6} - \frac{D^2}{6}\right)\right]}x^2$$

$$= -\frac{1}{6D}\left[1+\left(\frac{D}{6} - \frac{D^2}{6}\right)\right]^{-1} x^2$$

$$= -\frac{1}{6D}\left[1-\left(\frac{D}{6} - \frac{D^2}{6}\right)+\left(\frac{D}{6} - \frac{D^2}{6}\right)^2 - \ldots\ldots\ldots\right]x^2$$

$[\because \ (1+D)^{-1} = 1 - D + D^2 - D^3 + \ldots\ldots]$

$$= -\frac{1}{6D}\left[1-\frac{D}{6}+\frac{D^2}{6}+\frac{D^2}{36}+\text{terms of higher powers of } D\right]x^2$$

$$= -\frac{1}{6D}\left[1-\frac{D}{6}+\frac{7}{36}D^2+.........\right]x^2$$

$$=-\frac{1}{6D}\left[x^2-\frac{1}{6}Dx^2+\frac{7}{36}D^2x^2\right] \qquad \text{[Higher derivatives become zero]}$$

$$= -\frac{1}{6D}\left[x^2-\frac{2x}{6}+\frac{7}{18}\right]$$

$$= -\frac{1}{6}\left[\frac{x^3}{3}-\frac{1}{3}.\frac{x^2}{2}+\frac{7}{18}x\right] \qquad \left[\because \frac{1}{D}f(x)=\int f(x)dx\right]$$

$$= -\frac{1}{18}\left(x^3-\frac{x^2}{2}+\frac{7}{6}x\right)$$

Therefore the general solution of equation (1) is

$$y = C.F + P.I.$$

or $$y = c_1+c_2e^{-2x}+c_3e^{3x}-\frac{1}{18}\left(x^3-\frac{x^2}{2}+\frac{7}{6}x\right)$$

VIII If $X = e^{ax}V$ where V is a function of x, then $\frac{1}{f(D)}(e^{ax}V)=e^{ax}\frac{1}{f(D+a)}V$

Example 5 : Solve $(D^4-4D+4)y=e^{2x}\sin^2 x$

Solution : The given differential equation is $(D^4-4D+4)y=e^{2x}\sin^2 x$ ......(1)

$\therefore$ Auxiliary equation is $m^4-4m+4=0 \Rightarrow (m-2)^2=0 \Rightarrow m=2, 2$

$\therefore$ $C.F. = (c_1+c_2x)e^{2x}$

Now. $$P.I.= \frac{1}{D^2-4D+4}e^{2x}\sin^2 x = e^{2x}\frac{1}{(D+2)^2-4(D+2)+4}\sin^2 x$$

$$\left[\because \frac{1}{f(D)}e^{ax}V=e^{ax}\frac{1}{f\left(D+a\right)}V\right]$$

$$= e^{2x}\frac{1}{D^2+4D+4-4D-8+4}\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{e^{2x}}{2}\frac{1}{D^2}(1-\cos 2x)$$

$$= \frac{e^{2x}}{2}\iint(1-\cos 2x)dx$$

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$$= \frac{e^{2x}}{2}\int\left(x - \frac{\sin 2x}{2}\right)dx$$

$$= \frac{e^{2x}}{2}\left[\frac{x^2}{2} + \frac{\cos 2x}{4}\right] = \frac{e^{2x}}{8}\left[2x^2 + \cos 2x\right]$$

Therefore the complete solution is

$$y = C.F. + P.I.$$

or $$y = (c_1 + c_2 x)e^{2x} + \frac{e^{2x}}{8}\left[2x^2 + \cos 2x\right]$$

IX If $X = xV$, where V is a function of x, then $\frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V + \frac{d}{dD}\left[\frac{1}{f(D)}\right]V$

Example 6 : Solve $\frac{d^2y}{dx^2} + 4y = x\cos x$

Solution : The given differential equation is

$$\frac{d^2y}{dx^2} + 4y = x\cos x \qquad \text{......(1)}$$

Equation (1) can be written as $(D^2+4)y = x\cos x$

$\therefore$ Auxiliary equation is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

$\therefore$ $C.F. = c_1\cos 2x + c_2 \sin 2x$

Now, $P.I. = \frac{1}{D^2+4}x\cos x$

$$= x\frac{1}{D^2+4}\cos x + \frac{d}{dD}\left(\frac{1}{D^2+4}\right)\cos x \quad \left[\because \frac{1}{f(D)}xV = x\frac{1}{f(D)}V + \frac{d}{dD}\left(\frac{1}{f(D)}\right)V\right]$$

$$= x\frac{1}{-1+4}\cos x + \frac{-2D}{(D^2+4)^2}\cos x$$

$$= \frac{x\cos x}{3} - \frac{2D}{(-1+4)^2}\cos x$$

$$= \frac{x\cos x}{3} - \frac{2}{9}D\cos x = \frac{x\cos x}{3} + \frac{2}{9}\sin x$$

Therefore the complete solution is

$$y = C.F. + P.I.$$

or $$y = c_1\cos 2x + c_2\sin 2x + \frac{x\cos x}{3} + \frac{2}{9}\sin x$$

Exercise 3.2

Solve the following differential equations :

1. $\frac{d^2y}{dx^2}+\frac{dy}{dx}+9y=2\sinh 3x$

2. $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=e^{-4x}+5\cos 3x$

3. $\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}-6\frac{dy}{dx}=1+x^2$

4. $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=e^{2x}\cos^2 x$

5. $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=xe^x\sin 2x$

Answers

1. $y=(c_1+c_2x)e^{-3x}+\frac{e^{3x}}{36}-\frac{x^2e^{-3x}}{2}$

2. $y=(c_1+c_2x)e^{2x}+\frac{1}{36}e^{-4x}-\frac{5}{169}(12\sin 3x+5\cos 3x)$

3. $y=c_1+c_2e^{-2x}+c_3e^{3x}-\frac{1}{108}(6x^3-3x^2+25x)$

4. $y=(c_1+c_2x)e^{2x}+\frac{1}{8}e^{2x}(2x^2-\cos 2x)$

5. $y=(c_1+c_2x)e^x-\frac{e^x}{4}(x\sin 2x+\cos 2x)$

Def. Homogeneous Linear Differential Equation (or Cauchy-Euler Equation):

A linear differential equation of the form

$$x^n\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+......+a_ny=X$$

where $a_1, a_2,....,a_n$ are constants and X is either a constant or a function of x only, is called a homogeneous linear differential equation.

Method to solve homogeneous linear differential equation :

Consider the linear differential equation

$$x^n\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+......+a_ny=X \qquad ......(1)$$

To solve (1) we introduce a new independent variable z , such that

$$x=e^z \text{ or } z=\log x \text{ and } D\equiv\frac{d}{dz}$$

so that $\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{dy}{dz}\frac{1}{x}$ $\left[\because z=\log x \text{ so } \frac{dz}{dx}=\frac{1}{x}\right]$

or $x\frac{dy}{dx}=\frac{dy}{dz}=Dy$ ......(2)

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and $$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx}=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x^2}\frac{d^2y}{dz^2}=\frac{1}{x^2}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)$$

or $$x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}$$

$$=(D^2-D)y$$

$$=D(D-1)y \qquad \text{......(3)}$$

Similarly, $$x^3\frac{d^3y}{dx^3}=D(D-1)(D-2)y \qquad \text{......(4)}$$

In general, $$x^n\frac{d^ny}{dx^n}=D(D-1)(D-2)....(D-n+1)y \qquad \text{......(5)}$$

Substituting (2), (3), (4) etc. into (1), we get a *L.D.E.* with constant coefficients and is solvable for y in terms of z by the methods discussed in chapter 4.

Example 1 : Solve $x^2\frac{d^2y}{dx^2}+9x\frac{dy}{dx}+25y=x^2+\frac{1}{x}$.

Solution : The given differential equation is $x^2\frac{d^2y}{dx^2}+9x\frac{dy}{dx}+25y=x^2+\frac{1}{x}$ ......(1)

which is a homogeneous linear differential equation.

Let $e^z=x \Rightarrow z=\log x$

Thus , we get

$$x\frac{d}{dx}=D \text{ and } x^2\frac{d^2}{dx^2}=D(D-1) \text{ where } \frac{d}{dz}=D$$

Then equation (1) can be written as

$$[D(D-1)+9D+25]y=e^{2z}+e^{-z}$$

or $$(D^2+8D+25)y=e^{2z}+e^{-z}$$

$\therefore$ Auxiliary equation is $m^2+8m+25=0$

$$\therefore \qquad m=\frac{-8\pm\sqrt{64-100}}{2}=\frac{-8\pm 6i}{2}=-4\pm 3i$$

$\therefore \qquad C.F. = e^{-4z}(c_1 \cos 3z + c_2 \sin 3z)$

$$= x^{-4}[c_1 \cos 3(\log x) + c_2 \sin 3(\log x)] \qquad [\because \; e^{-z} = x^{-1}]$$

Here $\qquad P.I. = \dfrac{1}{D^2+8D+25}(e^{2z}+e^{-z})$

$$= \frac{1}{D^2+8D+25}e^{2z} + \frac{1}{D^2+8D+25}e^{-z}$$

$$= \frac{1}{4+16+25}e^{2z} + \frac{1}{1-8+25}e^{-z}$$

$$= \frac{1}{45}e^{2z} + \frac{1}{18}e^{-z} = \frac{1}{45}x^2 + \frac{1}{18}x^{-1} \qquad [\because e^z = x]$$

$\therefore$ The general solution is

$$y = C.F. + P.I.$$

or $\qquad y = x^{-4}[c_1 \cos 3(\log x) + c_2 \sin 3(\log x)] + \dfrac{1}{45}x^2 + \dfrac{1}{18}x^{-1}.$

Def. Legendre's Linear Differential Equation :

A differential equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + a_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ... + a_n y = X \qquad ......(1)$$

where $a, b, a_1, a_2, ..., a_n$ are constants and X is either a constant or a function of x only, is known as Legendre's linear equation. Such equations can also be reduced to *L.D.E.* with constant coefficients.

Let $ax+b = e^z$ or $z = \log(ax+b)$ and $D = \dfrac{d}{dz}$

Then $\qquad \dfrac{dy}{dx} = \dfrac{dy}{dz}\dfrac{dz}{dx} = \dfrac{a}{ax+b}\dfrac{dy}{dz}$

or $\qquad (ax+b)\dfrac{dy}{dx} = aDy$

Similarly $(ax+b)^2 \dfrac{d^2y}{dx^2} = a^2 D(D-1)y$

and $\qquad (ax+b)^3 \dfrac{d^3y}{dx^3} = a^3 D(D-1)(D-2)y$

In general $\qquad (ax+b)^n \dfrac{d^n y}{dx^n} = a^n D(D-1)(D-2)....(D-n+1)y$

By substituting these values in equation (1), we obtain a *L.D.E.* with constant coefficients, which can be solve by the methods discussed earlier.

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Example 2 : Solve $(3x+2)^2\frac{d^2y}{dx^2}+5(3x+2)\frac{dy}{dx}-3y=x^2+x+1$

Solution : The given differential equation is

$$(3x+2)^2\frac{d^2y}{dx^2}+5(3x+2)\frac{dy}{dx}-3y=x^2+x+1 \qquad \text{...... (1)}$$

which is a Legendre's linear differential equation.

Let $3x+2=e^z \quad \Rightarrow \quad x=\frac{e^z-2}{3}$ and $z=\log(3x+2)$

Thus , we get $\quad (3x+2)\frac{d}{dx}=3D$

$$(3x+2)^2\frac{d^2}{dx}=3^2D(D-1) \qquad \text{where} \qquad \frac{d}{dz}=D$$

Then equation (1) can be written as

$$\left(3^2D(D-1)+5.3D-3\right)y=\left(\frac{e^z-2}{3}\right)^2+\frac{e^z-2}{3}+1$$

or $\quad (9D^2-9D+15D-3)y=\frac{e^{2z}-4e^z+4}{9}+\frac{e^z-2}{3}+1$

$\Rightarrow \quad (9D^2+6D-3)y=\frac{1}{9}e^{2z}-\frac{e^z}{9}+\frac{7}{9}$

$\therefore$ Auxiliary equation is

$$9m^2+6m-3=0$$

or $\quad 3m^2+2m-1=0$

or $\quad (3m-1)(m+1)=0 \quad \Rightarrow \quad m=-1,\ \frac{1}{3}$

$\therefore \quad C.F. = c_1e^{-z}+c_2e^{\frac{1}{3}z}=c_1(3x+2)^{-1}+c_2(3x+2)^{\frac{1}{3}}$

Here $\quad P.I.=\frac{1}{9D^2+6D-3}\left(\frac{1}{9}e^{2z}-\frac{e^z}{9}+\frac{7}{9}\right).$

$$=\frac{1}{27}\left(\frac{1}{3D^2+2D-1}e^{2z}-\frac{1}{3D^2+2D-1}e^z+\frac{1}{3D^2+2D-1}7e^{0z}\right)$$

$$=\frac{1}{27}\left(\frac{1}{12+4-1}e^{2z}-\frac{1}{3+2-1}e^z+\frac{1}{-1}7e^{0z}\right)$$

$$= \frac{1}{27}\left(\frac{1}{15}e^{2z} - \frac{1}{4}e^{z} - 7\right)$$

$$= \frac{1}{27}\left(\frac{(3x+2)^2}{15} - \frac{(3x+2)}{4} - 7\right)$$

$$= \frac{(3x+2)^2}{405} - \frac{(3x+2)}{108} - \frac{7}{27}$$

$\therefore$ The general solution is

$$y = C.F + P.\,I.$$

or
$$y = c_1(3x+2)^{-1} + c_2(3x+2)^{\frac{1}{3}} + \frac{(3x+2)^2}{405} - \frac{(3x+2)}{108} - \frac{7}{27}$$

Exercise 3.3

Solve the following differential equations :

1. $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = x + \sin x$

2. $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{12\log x}{x^2}$

3. $(3x+2)^2\dfrac{d^2y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1$

Answers

1. $y = c_1x^{-1} + c_2x^{-2} + \dfrac{x}{6} - \dfrac{1}{x^2}\sin x$

2. $y = c_1 + c_2\log x + 2(\log x)^3$

3. $y = c_1(3x+2)^2 + c_2(3x+2)^{-2} + \dfrac{1}{108}\left[(3x+2)^2\log(3x+2) + 1\right]$

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Assignment-3

--------------------- S C Q ---------------------

1. The particular solution of the equation $y'\sin x = y\log y$ satisfying the initial condition $y\left(\frac{\pi}{2}\right) = e$ is

   1. $e^{\tan\left(\frac{x}{2}\right)}$ 2. $e^{\cot\left(\frac{x}{2}\right)}$
   3. $\log\tan\left(\frac{x}{2}\right)$ 4. $\log\cot\left(\frac{x}{2}\right)$

2. A particular solution of the differential equation $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + \frac{y}{4} = \frac{1}{\sqrt{x}}$ is

   1. $\frac{1}{2\sqrt{x}}$ 2. $\frac{\log x}{2\sqrt{x}}$
   3. $\frac{(\log x)^2}{2\sqrt{x}}$ 4. $\frac{(\log x)\sqrt{x}}{2}$

3. One particular solution of $y''' - y'' - y' + y = -e^x$ is a constant multiple of

   1. xe^{-x} 2. xe^{x}
   3. $x^2 e^{-x}$ 4. $x^2 e^{x}$

4. Let $y(x)$ be the solution of the initial value problem $y''' - y'' + 4y' - 4y = 0$, $y(0) = y'(0) = 2,\ y''(0) = 0$

   Then the value of $y\left(\frac{\pi}{2}\right)$ is

   1. $\frac{1}{5}\left(4e^{\frac{\pi}{2}} - 6\right)$ 2. $\frac{1}{5}\left(6e^{\frac{\pi}{2}} - 4\right)$
   3. $\frac{1}{5}\left(8e^{\frac{\pi}{2}} - 2\right)$ 4. $\frac{1}{5}\left(8e^{\frac{\pi}{2}} + 2\right)$

5. Let $y(x)$ be the solution of the initial value problem $x^2y'' + xy' + y = x,\ \ y(1) = y'(1) = 1$

   Then, the value of $y\left(e^{\frac{\pi}{2}}\right)$ is

   1. $\frac{1}{2}\left(1 - e^{\frac{\pi}{2}}\right)$ 2. $\frac{1}{2}\left(1 + e^{\frac{\pi}{2}}\right)$
   3. $\frac{1}{2} + \frac{\pi}{4}$ 4. $\frac{1}{2} - \frac{\pi}{4}$

6. If $y(x) = x$ is a solution of the differential equation $y'' - \left(\frac{2}{x^2} + \frac{1}{x}\right)(xy' - y) = 0$, $0 < x < \infty$, then its general solution is

   1. $\left(\alpha + \beta e^{-2x}\right)x$ 2. $\left(\alpha + 2\beta e^{2x}\right)x$
   3. $\alpha x + \beta e^{x}$ 4. $\left(\alpha e^{x} + \beta\right)x$

7. If $y = \phi(x)$ is a particular solution of $y'' + (\sin x)y' + 2y = e^x$ and $y = \psi(x)$ is a particular solution of $y'' + (\sin x)y' + 2y = \cos 2x$, then a particular solution of $y'' + (\sin x)y' + 2y = e^x + 2\sin^2 x$ is given by

   1. $\phi(x) - \psi(x) + \frac{1}{2}$
   2. $\psi(x) - \phi(x) + \frac{1}{2}$
   3. $\phi(x) - \psi(x) + 1$
   4. $\psi(x) - \phi(x) + 1$

8. Suppose $y_p(x) = x\cos 2x$ is a particular solution of $y'' + \alpha y = -4\sin 2x$. Then, the constant α equals

1. 1   2. –2   3. 2   4. 4

9. If $D = \frac{d}{dx}$, then the value of $\frac{1}{(xD+1)}(x^{-1})$ is

1. $\log x$ 2. $\frac{\log x}{x}$

3. $\frac{\log x}{x^2}$ 4. $\frac{\log x}{x^3}$

10. Determine the type of the following differential equation

$\frac{d^2y}{dx^2} + \sin(x+y) = \sin x$

1. linear, homogeneous
2. non linear, homogeneous
3. linear, non-homogeneous
4. non-linear, non-homogeneous

11. Linear combination of solution of an ordinary differential equation are also solutions, if the differential equation is

1. linear non-homogeneous
2. linear homogeneous
3. non linear homogeneous
4. non linear non homogeneous

12. Let $y(x)$ be a continuous solution of the initial value problem $y' + 2y = f(x)$,

$y(0) = 0$, where $f(x) = \begin{cases} 1 & , \quad 0 \le x \le 1 \\ 0 & , \quad x > 1 \end{cases}$

Then $y\left(\frac{3}{2}\right)$ is equal to

1. $\frac{\sinh(1)}{e^2}$ 2. $\frac{\cosh(1)}{e^2}$

3. $\frac{\sinh(1)}{e^3}$ 4. $\frac{\cosh(1)}{e^3}$

13. If $\frac{c_1 + c_2 \ln x}{x}$ is the general solution of the differential equation $x^2\frac{d^2y}{dx^2} + kx\frac{dy}{dx} + y = 0$ $x > 0$, then k equals

1. 3   2. –3   3. 2   4. –1

14. A non trivial solutions of $x^2y'' + xy' + 4y = 0$, $x > 0$ are

1. Bounded and non-periodic
2. unbounded and non-periodic
3. Bounded and periodic
4. unbounded and periodic

15. The solution $y(x)$ of the differential equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$ satisfying the conditions $y(0) = 4$, $\frac{dy}{dx}(0) = 8$ is

1. $4e^{2x}$ 2. $(16x+4)e^{-2x}$

3. $4e^{-2x}$ 4. $4e^{-2x} + 16x\,e^{2x}$

16. The general solution of $x^2\frac{d^2y}{dx^2} - 5x\frac{dy}{dx} + 9y = 0$ is

1. $(c_1 + c_2 x)e^{3x}$ 2. $(c_1 + c_2 \ln x)x^3$

3. $(c_1 + c_2 x)x^3$ 4. $(c_1 + c_2 \ln x)e^{x^3}$

17. All real solutions of the differential equation $y'' + 2ay' + by = \cos x$ (where a and b are real constants) are periodic if

1. $a = 1$ and $b = 0$ 2. $a = 0$ and $b = 1$
3. $a = 1$ and $b \ne 0$ 4. $a = 0$ and $b \ne 1$

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18. The differential equation whose linearly independent solutions are $\cos 2x,\ \sin 2x$ and e^{-x} is
    1. $(D^3+D^2+4D+4)y=0$
    2. $(D^3-D^2+4D-4)y=0$
    3. $(D^3+D^2-4D-4)y=0$
    4. $(D^3-D^2-4D+4)y=0$

19. The solution of the first order ODE $xy'=xy+x+y+1$
    1. $y=cx(e^x-1)$
    2. $y=cxe^x-1$
    3. $y=ce^x-x$
    4. $y=ce^x-x-1$

20. The general solution of the first order ODE $xy'+x^2y-y=0$ is
    1. $y(x)=xe^{-\frac{x^2}{2}}$
    2. $y(x)=e^{\frac{x^2}{2}}(x+c)$
    3. $y(x)=e^{\frac{x^2}{2}}-cx$
    4. $y(x)=cxe^{-\frac{x^2}{2}}$

21. The general solution of the first order ODE $xy'+2x^2y-x\,e^{-x^2}=0$ is
    1. $y(x)=e^{x^2}(x+c)$
    2. $y(x)=e^{-x^2}(x+c)$
    3. $y(x)=xe^{x^2}+c$
    4. $y(x)=x+c$

-------------------------- M C Q --------------------

1. Let $y:R\to R$ be a solution of the ordinary differential equation $2y''+3y'+y=e^{-3x}$, $x\in\mathbb{R}$, satisfying $\lim_{x\to\infty} e^x y(x)=0$. Then
    1. $\lim_{x\to\infty} e^{2x}y(x)=0$
    2. $y(0)=\frac{1}{10}$
    3. y is a bounded function on $\mathbb{R}$
    4. $y(1)=0$

Answers

----------------------- S C Q --------------------

1. 1	2. 3	3. 4
4. 3	5. 2	6. 4
7. 1	8. 4	9. 2
10. 4	11. 2	12. 1
13. 1	14. 3	15. 2
16. 2	17. 2	18. 1
19. 2	20. 4	21. 2

---------------------- M C Q --------------------

1. 1,2

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Chapter - 4

Method of Variation of Parameters :

Here we shall explain the method of finding the complete solution of a linear differential equation of second order whose complementary function is known.

Let $$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R \qquad ......(1)$$

be a linear differential equation of the second order where P, Q and R are functions of x.

Working Rule :

Step I : Put the equation in the standard form $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$ in which the co-efficient of $\frac{d^2y}{dx^2}$ must be unity.

Step II : Taking $R = 0$ in standard form , we get $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0$. Then find two independent solutions of this equation and call them y_1 and y_2. So, we have

$$C.F. = c_1'y_1+c_2'y_2 \ , \ c_1', c_2' \text{ being arbitrary constants.}$$

Step III : Consider the general solution of given equation, by replacing the arbitrary constants c_1' and c_2' by $u(x)$ and $v(x)$ in C.F. i.e., $y = u(x)y_1+v(x)y_2$

Step IV : Find $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$, $u(x) = -\int\frac{y_2R}{W}dx+c_1$ and $v(x)=\int\frac{y_1R}{W}dx+c_2$

Step V : Putting the values of $u(x)$ and $v(x)$ in $y = u(x)y_1 + v(x)y_2$, we get the required solution.

Example 1 : Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2}+4y=\tan 2x$

Solution : The given differential equation is $\frac{d^2y}{dx^2}+4y=\tan 2x$ ......(1)

Comparing (1) with $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$, we get $P=0,\ Q=4,\ R=\tan 2x$

The symbolic form of the equation (1) is

$$(D^2+4)y=\tan 2x \quad \text{where } D = \frac{d}{dx}$$

$\therefore$ *A.E.* is $m^2 + 4 = 0$

or $m^2 = -4 \quad \Rightarrow \quad m = \pm\, 2i$

$\therefore$ *C.F.* $= c_1' \cos 2x + c_2' \sin 2x$

Let the complete solution of differential equation (1) be

$$y = u(x)\cos 2x + v(x)\sin 2x \qquad \text{......(2)}$$

where $u(x)$ and $v(x)$ are unknown functions.

Let $\cos 2x = y_1$ and $\sin 2x = y_2$

Then $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$\therefore \quad = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 2$

Now $u(x) = -\int \dfrac{y_2 R}{W} dx + c_1$

$$= -\int \frac{(\sin 2x)\tan 2x}{2} dx + c_1$$

$$= -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x} dx + c_1$$

$$= -\frac{1}{2}\int \left(\frac{1-\cos^2 2x}{\cos 2x}\right) dx + c_1$$

$$= -\frac{1}{2}\int (\sec 2x - \cos 2x) dx + c_1$$

$$= -\frac{1}{2}\left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right] + c_1$$

$$= -\frac{1}{4}\left[\log(\sec 2x + \tan 2x) - \sin 2x\right] + c_1$$

and $v(x) = \int \dfrac{y_1 R}{W} dx + c_2$

$$= \int \frac{\cos 2x \ \tan 2x}{2} dx + c_2$$

$$= \frac{1}{2}\int \sin 2x \, dx + c_2$$

$$= -\frac{\cos 2x}{4} + c_2$$

From equation (2), we have

$$y = u(x)\cos 2x + v(x)\sin 2x$$

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$$= -\frac{1}{4}\left[\log(\sec 2x + \tan 2x) - \sin 2x\right]\cos 2x + c_1 \cos 2x - \frac{\cos 2x}{4}\sin 2x + c_2 \sin 2x$$

or $$y = -\frac{\cos 2x}{4}\left[\log(\sec 2x + \tan 2x) - \sin 2x + \sin 2x\right] + c_1 \cos 2x + c_2 \sin 2x$$

So the complete solution of (1) is

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{\cos 2x}{4}[\log(\sec 2x + \tan 2x)]$$

The Method Of Undetermined Coefficients :

Def. UC functions : A function is called *UC* function if it is either

(I) A function defined by one of the following

(i) x^n, n being a positive integer or zero.

(ii) e^{ax}, a being a non-zero constant.

(iii) $\sin(bx + c)$, where b and c are constants and $b \neq 0$.

(iv) $\cos(bx + c)$, where b and c are constants and $b \neq 0$.

or

(II) A function defined as a finite product of two or more function of these four types. For example x^2, e^{-3x}, $\sin(3x + \pi/4)$, $x^2 e^{3x}$, $x^3 \cos 2x$, $xe^{4x}\sin 3x$ are all *UC* function.

The method of undetermined coefficients (briefly called *UC* method) is applicable only when the non-homogeneous function is a finite linear combination of *UC* functions.

Def. *UC* set : Consider a *UC* function f. The set of functions consisting of f itself and all linearly independent *UC* functions of which the successive derivatives of f are either constant multiples or linear combinations is called the *UC* set of f. For example

(i) consider the function $f(x) = x^3$

then

$$f'(x) = 3x^2\ ,\ f''(x) = 6x,\ f'''(x) = 6\ ,......,\ f^n(x) = 0\ , n > 3$$

We see that all the successive derivatives of f are either multiples or linear combinations of the functions x^2, x , 1. Thus the *UC* set of x^3 is the set

$$S = \{x^3, x^2, x, 1\}$$

(ii) As another example, consider the function $f(x) = x^2 \sin x$

then $$f'(x) = 2x\sin x + x^2 \cos x$$

$$f''(x) = 2\sin x + 4x\cos x - x^2 \sin x$$

$$f'''(x) = 6\cos x - 6x\sin x - x^2\cos x \text{ and so on.}$$

We observe that no new type of function will be obtained from further differentiation. Thus all the derivatives of $f(x)$ are linear combination of $x^2\sin x,\ x^2\cos x,\ x\sin x,\ x\cos x,\ \sin x,\ \cos x$.

Thus UC set of $x^2\sin x$ is given by :

$$S = \{x^2\sin x,\ x^2\cos x,\ x\sin x,\ x\cos x,\ \sin x,\ \cos x\}$$

Here is the list of the UC sets of different UC functions.

Sr. No.	***UC* function**	***UC* set**
1.	x^n	$\{x^n,\ x^{n-1},\ x^{n-2},\ldots\ldots\ldots,\ x, 1\}$
2.	e^{ax}	$\{e^{ax}\}$
3.	$\sin bx$ or $\cos bx$	$\{\sin bx,\ \cos bx\}$
4.	$x^n\ e^{ax}$	$\{x^n e^{ax},\ x^{n-1}e^{ax},\ldots\ldots, xe^{ax}, e^{ax}\}$
5.	$x^n\sin bx$ or $x^n\cos bx$	$\{x^n\sin bx, x^n\cos bx, x^{n-1}\sin bx, x^{n-1}\cos bx,\ldots\ldots$ $\ldots\ldots\ldots,\ x\sin bx, x\cos bx,\ \sin bx,\ \cos bx\}$
6.	$e^{ax}\sin bx$ or $e^{ax}\cos bx$	$\{e^{ax}\sin bx,\ e^{ax}\cos bx\}$

Working Rule : We now outline the method of undetermined co-efficient for finding a particular integral y_p of the equation $a_0\dfrac{d^2y}{dx^2} + a_1\dfrac{dy}{dx} + a_2 y = f(x)$

where f is a finite linear combination.

i.e., $f = A_1u_1 + A_2u_2 + \ldots\ldots\ldots + A_m u_m$ of UC functions u_1, u_2,......., u_m where $A_1, A_2,\ldots\ldots\ldots, A_m$ are arbitrary constant. Assuming that the complementary function y_c has already been obtained, we proceed as follows :

Step I. For each of the UC functions $u_1, u_2,\ldots\ldots\ldots, u_m$, form the corresponding UC set, thus obtaining the respective UC set $S_1, S_2,\ \ldots., S_m$.

Step II. In case one of the UC sets , so formed, say S_j is identical with or completely contained in another, say S_k ; we omit the set S_j from further consideration, retaining the set S_k.

Step III. Now we consider the UC sets which remain after step (II).

In case one of these UC sets, say 'S_i' includes one or more members which are solutions of the corresponding homogeneous equation (i.e. appear in the complementary function), we multiply each member of S_i by the lowest positive integral power of x so that the resulting revised set contains no member which is a part of complementary function. We then replace S_i by this new revised set. This step is repeated for all UC sets one by one.

Step IV. Now, we are left with **(i)** Certain of the original *UC* sets, which are neither omitted in step (II) nor revised in step (III) and **(ii)** Certain revised set resulting form step (III).

Now form a linear combination of all the sets of these two categories, with unknown constant co-efficients (called undetermined coefficients).

Step V. Assuming that this linear combination is a particular solution of given equation we can find the unknown contstant by substituting the linear combination in the differential equation. This completes the method.

The following examples will illustrate the method more clearly.

Example 2 : Find the general solution using the method of undetermined co-efficent, $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^x - 10\sin x$

Solution : The corresponding homogeneous equation is $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$ and the complementary function is $y_c = c_1e^{3x} + c_2e^{-x}$

The non-homogeneous terms is the linear combination of the two *UC* functions e^x and sin$x$.

1. Form the *UC* set for each of these two functions. i.e. $S_1 = \{e^x\}$, $S_2 = \{\sin x, \cos x\}$
2. Note here, that neither of S_1 and S_2 is identical with nor included in the other, hence both are retained.
3. By examining the complementary function, we see that no element of S_1 and S_2 is a part of *C.F.* , so no revision is required.
4. Thus the original sets S_1 and S_2 remain as such and we form the linear combination, $Ae^x + B\sin x + C\cos x$ of the elements of S_1 and S_2 with the undetermined coefficients A , B , C
5. We take $y_p = Ae^x + B\sin x + C\cos x$ as a particular solution, then

$$y_p' = Ae^x + B\cos x - C\sin x$$

$$y_p'' = Ae^x - B\sin x - C\cos x$$

Substituting in the given differential equation, we have

$$(Ae^x - B\sin x - C\cos x) - 2(Ae^x + B\cos x - C\sin x) - 3(Ae^x + B\sin x + C\cos x) = 2e^x - 10\sin x$$

or $$-4Ae^x + (-4B + 2C)\sin x + (-4C - 2B)\cos x = 2e^x - 10\sin x$$

Equating coefficients of like terms, we obtain

$$-4A = 2\ , \qquad -4B + 2C = -10\ , \qquad -4C - 2B = 0$$

which, on solving, give $\quad A = -1/2\ ,\ B = 2\ ,\ C = -1$

and hence, we obtain the particular integral

$$y_p = -\frac{1}{2}e^x + 2\sin x - \cos x$$

Thus the general solution of given differential equation is

$$y = y_c + y_p$$

$$= c_1 e^{3x} + c_2 e^{-x} - \frac{1}{2}e^x + 2\sin x - \cos x$$

Exercise 4.1

Apply the method of variation of parameters to solve the following differential equations :

1. $\dfrac{d^2y}{dx^2} - y = \dfrac{2}{1+e^x}$
2. $\dfrac{d^2y}{dx^2} + a^2 y = \sec ax$

Using the method of undetermined coefficients find the general solution of the following differential equation.

3. $y'' - 3y' + 2y = 14\sin 2x - 18\cos 2x$
4. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = 10e^{2x} - 18e^{3x} - 6x - 11$

Answers

1. $y = c_1 e^x + c_2 e^{-x} + e^x \log\left(\dfrac{e^x+1}{e^x}\right) - 1 - e^{-x}\log(e^x+1)$
2. $y = c_1 \cos ax + c_2 \sin ax + \dfrac{\cos ax \log\cos ax}{a^2} + \dfrac{x}{a}\sin ax$
3. $y = c_1 e^x + c_2 e^{2x} + 2\sin 2x + 3\cos 2x$
4. $y = c_1 e^{2x} + c_2 e^{-3x} + 2x e^{2x} - 3e^{3x} + x + 2$

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Assignment-4

--------------------- S C Q ---------------------

1. Solving by variation of parameter $y''-2y'+y=e^x\log x$, then the value of wronskian is

   1. e^{2x} 2. 2
   3. e^{-2x} 4. –2

2. For $\frac{d^2y}{dx^2}+4y=\tan 2x$ solving by variation of parameters. The value of wronskian W is

   1. 1 2. 2 3. 3 4. 4

3. Solving by variation of parameter for the equation $y''+y=\sec x$, the value of wronskian is

   1. 1 2. 2 3. 3 4. 4

4. Using the method of variation of parameters for the particular solution to the differential equation $y''+4y=\frac{3}{\sin 2x}$, $0<x<\frac{\pi}{2}$

   1. $\frac{3}{4}\sin 2x\log\cos 2x-\frac{3}{4}\cos 2x$
   2. $\frac{3}{2}\sin 2x\log\cos 2x-\frac{3}{4}\cos 2x$
   3. $\frac{3}{2}\sin 2x\log\sin 2x-\frac{3}{2}x\cos 2x$
   4. $\frac{3}{4}\sin 2x\log\sin 2x-\frac{3}{2}x\cos 2x$

5. Which of the following are not UC function.

   1. x^5 2. x
   3. $\tan x$ 4. $\sin x$

6. A UC set of $e^{2x}\sin 3x$ is

   1. $\{e^{2x}\}$
   2. $\{e^{2x}\sin 3x\}$
   3. $\{e^{2x}\cos 3x\}$
   4. $\{e^{2x}\sin 3x, e^{2x}\cos 3x\}$

7. Using the method of undetermined coefficients, find the general solution of $y''-3y'+2y=14\sin 2x-18\cos 2x$

   1. $y-c_1e^{x}+c_2e^{-x}+2\sin 2x+3\cos 2x$
   2. $y-c_1e^{x}+c_2e^{2x}+2\sin 2x+3\cos 2x$
   3. $y=c_1e^{2x}+c_2e^{-2x}+2\sin 2x+3\cos 2x$
   4. $y=c_1e^{x}+c_2e^{2x}+\sin 2x+3\cos 2x$

Answers

---------------------- S C Q --------------------

1. 1	2. 2	3. 1
4. 4	5. 3	6. 4
7. 2		

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Chapter - 5

Second order differential equations

Def. Linearly dependent function : Let $f_1, f_2, ..., f_n$ are real function defined on an interval I, then $f_1, f_2, ..., f_n$ are said to be L.D. over I if, there exist real no. $c_1, c_2, ..., c_n$ (not all zero) such that

$$c_1 f_1(x) + c_2 f_2(x) + ... + c_n f_n(x) = 0, \quad \forall x \in I$$

Def. Linearly Independent function : Let $f_1, f_2, ..., f_n$ are real function defined on an interval I, then $f_1, f_2, ..., f_n$ are said to be L.I. over I if, whenever $c_1 f_1(x) + c_2 f_2(x) + ... + c_n f_n(x) = 0, \forall x \in I$ we must have $c_1 = c_2 = ... = c_n = 0$

Result 1 : Let *f*, *g* are any function defined on an interval I, then *f* and *g* are L.D. over I iff there exist a constant *c* such that either $f(x) = c\, g(x)$ or $g(x) = c\, f(x), \quad \forall x \in I$

Result 2 : Suppose $f_1, f_2, ..., f_n$ are defined on an interval I, and J be any interval s.t $J \subseteq I$, then we have

(i) If $f_1, f_2, ..., f_n$ are L.D. on I then $f_1, f_2, ..., f_n$ are L.D. on J.

(ii) If $f_1, f_2, ..., f_n$ are L.I. on J then $f_1, f_2, ..., f_n$ are L.I. on I.

(iii) If $f_1, f_2, ..., f_n$ are L.I. on I then $f_1, f_2, ..., f_n$ may or may not L.I. on J.

(iv) If $f_1, f_2, ..., f_n$ are L.D. on J then $f_1, f_2, ..., f_n$ may or may not L.D. on I.

Exercise 5.1

1. Which of the following functions are L.D. / L.I. on $\mathbb{R}$:

(i) $\{1, x\}$	(ii) $\{1, x^2\}$	(iii) $\{x, x^2\}$	(iv) $\{x^m, x^n\}$, $m \neq n$
(v) $\{1, \sin x\}$	(vi) $\{x, \sin x\}$	(vii) $\{x^2, \sin x\}$	(viii) $\{1, \cos x\}$
(ix) $\{x, \cos x\}$	(x) $\{x^2, \cos x\}$	(xi) $\{\sin x, \cos x\}$	(xii) $\{\sin x, \sin 2x\}$
(xiii) $\{\sin x, 2\sin x\}$	(xiv) $\{\sin x, -\sin x\}$	(xv) $\{\cos x, -\cos 3x\}$	(xvi) $\{\cos x, -\cos 2x\}$
(xvii) $\{1, e^x\}$	(xviii) $\{x, e^x\}$	(xix) $\{\sin x, e^x\}$	(xx) $\{\cos x, e^x\}$
(xxi) $\{e^x, e^{-x}\}$	(xxii) $\{\sin x, x\sin x\}$	(xxiii) $\{e^x, xe^x\}$	(xxiv) $\{\cos x, x^2 \cos x\}$

(xxv) $\{1, \sinh x\}$ (xxvi) $\{1, \cosh x\}$ (xxvii) $\{x, \sinh x\}$ (xxviii) $\{x, \cosh x\}$

(xxix) $\{e^x, \sinh x\}$ (xxx) $\{e^x, \cosh x\}$ (xxxi) $\{\sin x, \sinh x\}$ (xxxii) $\{\cos x, \cosh x\}$

(xxxiii) $\{\sin x, \cosh x\}$ (xxxiv) $\{\sinh x, \cosh x\}$

2. Find the interval on which the following functions are L.D. and L.I.

(i) $f(x) = x^2$, $g(x) = \begin{cases} 2x^2, & -\infty < x < 0 \\ 3x^2, & 0 \le x < \infty \end{cases}$ (ii) $f(x) = \sin x$, $g(x) = \begin{cases} -\sin x, & -\infty < x < 0 \\ \sin x, & 0 \le x < \infty \end{cases}$

(iii) $f(x) = e^x$, $g(x) = \begin{cases} e^x, & -\infty < x < 0 \\ -e^x, & 0 \le x < \infty \end{cases}$

3. Find the interval on which the following function are L.D. / L.I.

(i) $f(x) = x, g(x) = |x|$ (ii) $f(x) = x^2, g(x) = |x^2|$ (iii) $f(x) = x^3, g(x) = |x^3|$

(iv) $f(x) = x^2, g(x) = x|x|$ (v) $f(x) = x^3, g(x) = x^2|x|$ (vi) $f(x) = x^4, g(x) = x^3|x|$

4. Which of the following functions are L.D. / L.I. on $\mathbb{R}$

(i) $1, x, x^2$ (ii) x^2, x^3, x^4 (iii) $1, \sin x, \cos x$

(iv) $\mathrm{e}^x, \sin x, \cos x$ (v) $\sinh x, \cosh x, \cos x$ (vi) $\sin x, \cos x, \sin\left(x + \frac{\pi}{3}\right)$

(vii) $\sin x, \cos x, \sin 2x$ (viii) e^x, e^{2x}, e^{-x} (ix) $e^x, e^{-x}, \sinh x$

(x) $e^x, e^{-x}, \cosh x$ (xi) $2\sin x, 3\cos x, \cos\left(x + \frac{\pi}{4}\right)$ (xii) $1, x, |x|$

Answers

1. (i) L.I. (ii) L.I. (iii) L.I. (iv) L.I. (v) L.I. (vi) L.I. (vii) L.I.
(viii) L.I. (ix) L.I. (x) L.I. (xi) L.I. (xii) L.I. (xiii) L.D. (xiv) L.D.
(xv) L.I. (xvi) L.I. (xvii) L.I. (xviii) L.I. (xix) L.I. (xx) L.I. (xxi) L.I.
(xxii) L.I. (xxiii) L.I. (xxiv) L.I. (xxv) L.I. (xxvi) L.I. (xxvii) L.I. (xxviii) L.I.
(xxix) L.I. (xxx) L.I. (xxxi) L.I. (xxxii) L.I. (xxxiii) L.I. (xxxiv) L.I.

2. (i) L.I. on $\mathbb{R}$ and L.D. on $(-\infty, 0)$ and $[0, \infty)$
(ii) L.I. on $\mathbb{R}$ and L.D. on $(-\infty, 0]$ and $[0, \infty)$
(iii) L.I. on $\mathbb{R}$ and L.D. on $(-\infty, 0]$ and $[0, \infty)$

3. (i) L.I. on $\mathbb{R}$ and L.D. on $(-\infty, 0]$ and $[0, \infty)$
(ii) L.D. on $\mathbb{R}$
(iii) L.I. on $\mathbb{R}$ and L.D. on $(-\infty, 0]$ and $[0, \infty)$
(iv) L.I. on $\mathbb{R}$ and L.D. on $(-\infty, 0]$ and $[0, \infty)$

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(v) L.I. on $\mathbb{R}$ and L.D. on $(-\infty,0]$ and $[0,\infty)$

(vi) L.I. on $\mathbb{R}$ and L.D. on $(-\infty,0]$ and $[0,\infty)$

4. (i) L.I. (ii) L.I. (iii) L.I. (iv) L.I. (v) L.I. (vi) L.D. (vii) L.I.
(viii) L.I. (ix) L.D. (x) L.D. (xi) L.D. (xii) L.I.

Wronskian theory

Def. Linear O.D.E. of order *n* : $a_0(x)\frac{d^n y}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+...+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=F(x)$

If $F(x)=0$ then equation is homogeneous and if $F(x)\neq 0$ then equation is non–homogeneous.

Linear : In any term there should not be two or more *y*'s multiply together. In other words, in any term there should not be product of two or more terms of the set $\left\{y,\frac{dy}{dx},........,\frac{d^n y}{dx^n}\right\}$.

e.g, $y.\frac{dy}{dx},\left(\frac{dy}{dx}\right)^2,y^2,y^3,y.\frac{d^2y}{dx^2}$ etc. should not be present .

Result 3 : FEUT : Consider the linear O.D.E. of order *n*,

$$a_0(x)\frac{d^n y}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+...+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=F(x) \qquad ......(1)$$

Requirements :

(i) Continuity requirement : The functions $a_0(x),a_1(x),...,a_n(x)$ and $F(x)$ are continuous on an interval *I*.

(ii) Non zero requirement : $a_0(x)\neq 0,\ \forall x\in I$

Conclusion : Let $c_0,c_1,c_2,...,c_{n-1}$ are any real constant and $x_0\in I$ be any point then exist a unique solution *f* of (1) s.t. $f(x_0)=c_0,\ f'(x_0)=c_1...,\ f^{n-1}(x_0)=c_{n-1}$. Moreover this solution *f* is defined over the entire interval *I*.

Illustration : If we take $x^2\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}+x^3y=e^x$ on $[1,10]$, then both requirement are fulfilled so FEUT is applicable. But if we take the interval $[-1,1]$, then non–zero requirement is not fulfilled and so FEUT is not applicable.

Def. Wronskian : Let $f_1, f_2,..., f_n$ be any *n* function defined on an interval *I*. Then the determinant

$$W(f_1, f_2,..., f_n)(x)=\begin{vmatrix} f_1(x) & f_2(x) & ... & f_n(x) \\ f_1'(x) & f_2'(x) & ... & f_n'(x) \\ ... & ... & ... & ... \\ f_1^{(n-1)}(x) & f_2^{(n-1)} & ... & f_n^{(n-1)}(x) \end{vmatrix}$$ is called the Wronskian of $f_1, f_2,..., f_n$ at the point *x*.

In particular, $W(f_1, f_2)(x)=\begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix}=f_1(x)\, f_2'(x)-f_2(x)\, f_1'(x)$

Result 4 : In all the following results we shall consider the linear homogenous ODE

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)\,y=0 \qquad ....(1)$$

where both the requirements are fulfilled (continuity and non zero) on an interval *I*.

Results :

5. Abel's – Liouville's formula : Suppose f_1, f_2 be any two solutions of (1) and $x_0 \in I$ be any point then $W(x)=W(f_1, f_2)(x)=W(f_1, f_2)(x_0)\exp\left(-\int_{x_0}^{x}\frac{a_1(t)}{a_0(t)}dt\right), \forall\, x\in I$

Remark : By this formula, the value of wronskian can be calculated at the general point if it is given at a particular point x_0.

6. Wronskian of any two solutions of (1) is either identically zero on *I* or never zero on *I*.
Note : This result is only for solutions of (1).

7. Two solutions f_1 and f_2 of (1) are L.D. over *I* iff $W(f_1, f_2)(x)=0, \forall\, x\in I$

8. Two solutions f_1 and f_2 of (1) are L.I. over *I* iff $W(f_1, f_2)(x)\neq 0, \forall\, x\in I$

9. Let f_1, f_2 be any two function defined on *I* such that $W(f_1, f_2)(x)\neq 0$, for some $x\in I$ then, f_1, f_2 are L.I

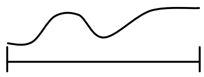
10. If $W(f_1, f_2)(x)=0, \forall\, x\in I$, then f_1, f_2 are may or may not be L.D.

e.g, Consider the functions $\{x, 2x\}$ on $\mathbb{R}$. Then $W(x,2x)=\begin{vmatrix} x & 2x \\ 1 & 2 \end{vmatrix}=0$ and function are clearly L.D.

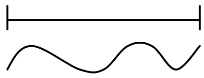
Again consider the functions $\{x^3, |x|^3\}$ on $\mathbb{R}$, then it can be verified that Wronskian is zero but functions are L.I.

11. Let f_1, f_2 are any two solution of (1) then $W(f_1, f_2)(x)$ keeps a constant sign on *I*, if it is non zero.

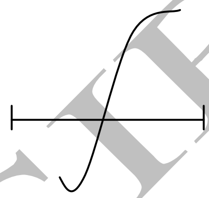
Remark : Graphs of Wronskian :



possible



possible



not possible

12. Let f_1, f_2 be any two solution of (1) and they have a common zero in interval *I* then f_1 and f_2 are L.D.

13. Contrapositive of above result : If two solution of (1) are L.I. then they can not have a common zero in *I*.

14. Let f_1, f_2 are any two solution of (1) and $x_0 \in I$ be any point. If f_1 and f_2 has a local maxima or local minima at x_0 then f_1 and f_2 are L.D.

Proof : By the given condition it is clear that $f_1'(x_0) = 0 = f_2'(x_0)$ and so

$$W(f_1, f_2)(x_0) = \begin{vmatrix} f_1(x_0) & f_2(x_0) \\ 0 & 0 \end{vmatrix} = 0$$

15. Contrapositive : If the solutions f_1 and f_2 of (1) are L.I. then both of them can not have a local maxima or a local minima at a same point.

16. Let f_1 and f_2 be any two solutions of (1) s.t. $f_2(x) \neq 0, \forall x \in I$ then $\frac{d}{dx}\left(\frac{f_1}{f_2}\right)(x) = \frac{-W(f_1, f_2)(x)}{f_2^2(x)}$

Proof : $\frac{d}{dx}\left(\frac{f_1}{f_2}\right)(x) = \frac{f_2(x) f_1'(x) - f_1(x) f_2'(x)}{f_2^2(x)} = \frac{-W(f_1, f_2)(x)}{f_2^2(x)}$

17. Let f_1, f_2 be any two solutions of (1) s.t. $f_2(x) \neq 0, \forall x \in I$, then $\frac{f_1}{f_2}$ is a monotonic function.

Proof : We know that $\frac{d}{dx}\left(\frac{f_1}{f_2}\right)(x) = \frac{-W(f_1, f_2)(x)}{f_2^2(x)}$

Case (i): If $W(f_1, f_2) \geq 0$ on I then $\frac{d}{dx}\left(\frac{f_1}{f_2}\right)(x) \leq 0$ for all $x \in I$ $\Rightarrow$ $\frac{f_1}{f_2}$ is decreasing function on I

Case (ii): If $W(f_1, f_2) \leq 0$ on I then $\frac{d}{dx}\left(\frac{f_1}{f_2}\right)(x) \geq 0$ for all $x \in I$ $\Rightarrow$ $\frac{f_1}{f_2}$ is increasing on I

18. The collection of all solutions of a linear homogeneous ODE of order *n* forms a vector space of dim *n*.

Illustration : $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$, General solution = $c_1e^{2x} + c_2e^{3x}$, V is a vector space of dim = 2,

Basis = $\{e^{2x}, e^{3x}\}$

19. Maximum number of L.I. solutions of a linear homogeneous ODE of order *n* is *n*.

20. Linear combination of solutions of a homogeneous ODE is again a solution.

21. Linear combination of solutions of a non homogeneous ODE is not always a solution.

22. Let $a_0(x)\frac{d^ny}{dx^n} + ... + a_n(x)y = F(x)$ ......(i) and $a_0(x)\frac{d^ny}{dx^n} + ... + a_n(x)y = 0$ ......(ii)

Let *f* be a solution of (i) and *g* be a solution of (ii) then $f + g$ is a solution of (i)

23. If $f_1, f_2, ..., f_n$ are solutions of (i). Then their convex combination is also a solution of (i)

24. The collection of all solutions of non homogeneous ODE does not form a vector space.

25. Let $\{f_1, f_2\}$ be a set of L.I. solutions of (1) and $\{g_1, g_2\}$ be another set of L.I. solution of (1) then , there exist a constant $c \neq 0$ s.t. $W(f_1, f_2)(x) = cW(g_1, g_2)(x)$ for all $x \in I$.

Proof : If $\{f_1, f_2\}$ be one set of L.I solutions of (1) and $\{g_1, g_2\}$ be another set of L.I. solutions of (1).

Then g_1 and g_2 are also the linear combination of f_1 and f_2, so result is proved.

26. Let f_1 and f_2 are two L.I. solution of (1) and $x_0 \in I$ be any point such that $f_1''(x_0) = f_2''(x_0) = 0$ then $a_1(x_0) = a_2(x_0) = 0$.

Remark : Basis is also called fundamental set.

27. Let *f* and *g* be a fundamental set of (1) over *I* then $\{af + bg, cf + dg\}$ is a fundamental set iff

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$

or

Let *f* and *g* are L.I solution of (1) , then $af + bg, cf + dg$ are also L.I. solution of (1) iff $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$

28. If $f(x)$ is any non – trivial solution of (1) then $f(x)$ cannot have a multiple zero in *I*.

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or

If $f(x)$ is a solution of (1) and there is a point $x_0 \in I$ s.t $f(x)=(x-x_0)^2 g(x)$ s.t. $g(x_0)\neq 0$ then $f(x)=0$ for all $x\in I$ i.e. f is a trivial solution.

Remark : The above result is also valid if the power 2 is replaced by 3, 4, 5, ...

Exercise 5.2

1. Find the Wronskian of the following :

(i) $\{1, x, x^2\}$ (ii) $\{1, \sin x, \cos x\}$ (iii) $\{1, x, e^x, \sin x, \cos x\}$

(iv) $\{1, x, e^x, \sin x, \cos x, \sinh x\}$ (v) $\{1, x, e^x, \sin x, \cos x, \cosh x\}$ (vi) $\{e^x, \sinh x, \cosh x\}$

2. Let y_1 and y_2 be two solutions of the problem $y''(t) + ay'(t) + by(t) = 0,\ t \in \mathbb{R}$ $y(0) = 0,$ where a and b are real constants. Let W be the wronskian of y_1 and y_2. Then find $W(t),$ for all $t \in \mathbb{R}$.

3. Let $y_1(x)$ and $y_2(x)$ form a complete set of solutions to the differential equation $y'' - 2xy' + \sin\left(e^{2x^2}\right) y = 0,\ \ x \in [0,1]$ with $y_1(0) = 0,\ y_1'(0) = 1,\ y_2(0) = 1,\ y_2'(0) = 1$. Then find the wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ at $x = 1$.

4. Let P, Q be continuous real valued functions defined on $[-1,1] \to \mathbb{R},\ \ i = 1,2$ be solutions of the ODE : $\dfrac{d^2u}{dx^2} + P(x)\dfrac{du}{dx} + Q(x)u = 0,\ \ x \in [-1,1]$ satisfying $u_1 \geq 0,\ u_2 \leq 0$ and $u_1(0) = 0,\ u_2(0) = 0$. Let W denote the wronskian of u_1 and u_2. Then find W. Also is u_1 and u_2 are linearly dependent or not.

5. Let P be continuous function on $\mathbb{R}$ and W the wronskian of two linearly independent solutions y_1 and y_2 of the ODE : $\dfrac{d^2y}{dx^2} + (1+x^2)\dfrac{dy}{dx} + P(x)y = 0,\ \ x \in \mathbb{R}$. Let $W(1) = a,\ W(2) = b,\ W(3) = c$. Then find a, b, c.

Answers

1. (i) 2 (ii) –1 (iii) $-2e^x$ (iv) –4 (v) 4 (vi) 0

2. $W(t) = 0,\ \forall\, t \in \mathbb{R}$ 3. $-e$ 4. $W = 0,$ L.D. 5. $a > b > c$ or $a < b < c$

Strum Theory

Def. Consider the second order homogeneous linear differential equation :

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=0 \qquad ......(1)$$

where a_0 has a continuous second derivative, a_1 has a continuous first derivative, a_2 is continuous, and $a_0(x)\neq 0$ on $a\le x\le b$. The adjoint equation to equation (1) is defined as

$$\frac{d^2}{dx^2}\left[a_0(x)y\right]-\frac{d}{dx}\left[a_1(x)y\right]+a_2(x)y=0. \qquad ......(2)$$

Remark : After taking the indicated derivatives in equation (2), we obtain

$$a_0(x)\frac{d^2y}{dx^2}+\left[2a_0'(x)-a_1(x)\right]\frac{dy}{dx}+\left[a_0''(x)-a_1'(x)+a_2(x)\right]y=0 \qquad ......(3)$$

where the primes denote differentiation with respect to *x*.

Def. The second order homogeneous linear differential equation $a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=0$ is called self adjoint if it is identical with its adjoint equation.

Theorem 1 : A necessary and sufficient condition for equation (1) to be self adjoint is that $\frac{d}{dx}\left[a_0(x)\right]=a_1(x)$ on $a\le x\le b.$

Corollary : If the equation $a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=0$ is self adjoint, then it can be written in the form $\frac{d}{dx}\left[a_0(x)\frac{dy}{dx}\right]+a_2(x)y=0$.

Proof : Since the given equation is self adjoint, so by Theorem 1, we have $a_0'(x)=a_1(x)$. Thus the given equation may be written as $a_0(x)\frac{d^2y}{dx^2}+a_0'(x)\frac{dy}{dx}+a_2(x)y=0$ or

$$\frac{d}{dx}\left[a_0(x)\frac{dy}{dx}\right]+a_2(x)y=0.$$

Theorem 2 : Let the coefficients a_0, a_1 and a_2 in the differential equation

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=0 \qquad ......(1)$$

are continuous on $a\le x\le b$ and $a_0(x)\neq 0$ on $a\le x\le b$.

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Then equation (1) can be transformed into the equivalent self adjoint equation $\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right]+Q(x)y=0$

on $a\le x\le b$, by multiplication throughout by the factor $\frac{1}{a_0(x)}\exp\left[\int\frac{a_1(x)}{a_0(x)}dx\right]$

Here, $P(x)=\exp\left[\int\frac{a_1(x)}{a_0(x)}dx\right]$, $Q(x)=\frac{a_2(x)}{a_0(x)}\exp\left[\int\frac{a_1(x)}{a_0(x)}dx\right]$.

Example : Consider the equation $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0$.

Here $a_0(x)=x^2, a_1(x)=-2x, a_2(x)=2.$ Since $a_0'(x)=2x\ne-2x=a_1(x)$, so given equation is not self adjoint. Let us form the factor $\frac{1}{a_0(x)}\exp\left[\int\frac{a_1(x)}{a_0(x)}dx\right]$ for this equation.

We have $\frac{1}{a_0(x)}\exp\left[\int\frac{a_1(x)}{a_0(x)}dx\right]=\frac{1}{x^2}\exp\left[\int\frac{-2x}{x^2}dx\right]=\frac{1}{x^4}$.

Multiplying equation $x^2\frac{d^2y}{dx^2}-2t\frac{dy}{dx}+2x=0$ by $\frac{1}{x^4}$ on any interval $a\le x\le b$ which does not include x=0, we obtain $\frac{1}{x^2}\frac{d^2y}{dx^2}-\frac{2}{x^3}\frac{dy}{dx}+\frac{2}{x^4}y=0$.

Since $\frac{d}{dx}\left(\frac{1}{x^2}\right)=-\frac{2}{x^3}$, this equation is self adjoint and may be written in the form $\frac{d}{dx}\left[\frac{1}{x^2}\frac{dy}{dx}\right]+\frac{2}{x^4}y=0.$

Theorem 3 : Let *f* be a solution of $\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right]+Q(x)y=0$ having first derivative f' on $a\le x\le b$. If *f* has an infinite number of zeros on $a\le x\le b$, then $f(x)=0$ for all *x* on $a\le x\le b$.

Theorem 4 : Abel's Formula : Let *f* and *g* be any two solutions of $\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right]+Q(x)y=0$ on the interval $a\le x\le b$. Then $P(x)\left[f(x)g'(x)-f'(x)g(x)\right]=k$, for all *x* on $a\le x\le b$, where *k* is a constant.

Theorem 5 : Let *f* and *g* be two solutions of $\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right]+Q(x)y=0$ s.t. *f* and *g* have a common zero on $a\le x\le b$. Then *f* and *g* are linearly dependent on $a\le x\le b$.

Theorem 6 : Let f and g be nontrivial linearly dependent solutions of equation $\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right]+Q(x)y=0$ on $a\le x\le b$, and suppose $f(x_0)=0,$ where x_0 is s.t. $a\le x\le b$. Then $g(x_0)=0$.

Theorem 7 : Sturm Separation Theorem : Let f and g be real linearly independent solutions of

$$\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right]+Q(x)y=0 \text{ on the interval } a\le x\le b.$$

Between any two consecutive zeros of f there is exactly one zero of g.

Theorem 8 : Sturm's Fundamental Comparison Theorem : On the interval $a\le x\le b$, Let ϕ_1 be a real solution of $\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right]+Q_1(x)y=0$. Let ϕ_2 be a real solution of $\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right]+Q_2(x)y=0$. Let P have a continuous derivative and be s.t. $P(x)>0$ and let Q_1 and Q_2 be continuous and s.t. $Q_2(x)> Q_1(x)$. If x_1 and x_2 are successive zeros of ϕ_1 on $[a,b]$, then ϕ_2 has atleast one zero at some point of the open interval $x_1<x<x_2$.

Exercise 5.3

1. Find the adjoint equation to each of the following equations :

(i) $x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+3y=0$ (ii) $(2x+1)\frac{d^2y}{dx^2}+x^3\frac{dy}{dx}+y=0$

(iii) $x^2\frac{d^2y}{dx^2}+(2x^3+7x)\frac{dy}{dx}+(8x^2+8)y=0$ (iv) $x^3\frac{d^2y}{dx^2}-(x^3+2x^2-x)\frac{dy}{dx}+(x^2+x-1)y=0$

2. Show that the adjoint equation of the adjoint equation of the equation $a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=0$ is the original equation itself.

3. Show that each of the following equations is self adjoint and write each in the form $\frac{d}{dx}\left[a_0(x)\frac{dy}{dx}\right]+a_2(x)y=0$.

(i) $x^3\frac{d^2y}{dx^2}+3x^2\frac{dy}{dx}+y=0$ (ii) $\sin x\frac{d^2y}{dx^2}+\cos x\frac{dy}{dx}+2y=0$ (iii) $\left(\frac{x+1}{x}\right)\frac{d^2y}{dx^2}-\frac{1}{x^2}\frac{dy}{dx}+\frac{1}{x^3}y=0$

4. Transform each of the following equations into an equivalent self adjoint equation :

(i) $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$ (ii) $(x^4+x^2)\frac{d^2y}{dx^2}+2x^3\frac{dy}{dx}+3x=0$

(iii) $\frac{d^2y}{dx^2}-\tan x\frac{dy}{dx}+y=0$ (iv) $f(x)\frac{d^2y}{dx^2}+g(x)y=0$

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5. Let $x:[0,3\pi]\to\mathbb{R}$ be a nonzero solution of the ODE $x''(t)+e^{t^2}x(t)=0,$ for $t\in[0,3\pi]$. Then find the cardinality of the set $\{t\in[0,3\pi]:x(t)=0\}$.

Answers

1. (i) $x^2y''+xy'+2y=0$ (ii) $(2x+1)y''+(4-x^3)y'+(1-3x^2)y=0$

   (iii) $x^2y''+(-3x-2x^3)y'+(2x^2+3)y=0$ (iv) $x^3y''+(x^3+8x^2-x)y'+(4x^2+11x-2)y=0$

3. (i) $\frac{d}{dx}\left[x^3\frac{dy}{dx}\right]+y=0$ (ii) $\frac{d}{dx}\left[\sin x\frac{dy}{dx}\right]+2y=0$

   (iii) $\frac{d}{dx}\left[\frac{(x+1)}{x}\frac{dy}{dx}\right]+\frac{1}{x^3}y=0$

4. (i) $\frac{d}{dx}\left[x\frac{dy}{dx}\right]+\frac{1}{x}y=0$ (ii) $\frac{d}{dx}\left[(x^2+1)\frac{dy}{dx}\right]+\frac{3}{x^2}y=0$

   (iii) $\frac{d}{dx}\left[\cos x\frac{dy}{dx}\right]+\cos x\cdot y=0$ (iv) $\frac{d}{dx}\left[\exp\left(\int\frac{g(x)}{f(x)}dx\right)\frac{dy}{dx}\right]=0$

5. ≥ 3

Sturm Liouville Boundary Value Problems

These problems represents a class of linear boundary value problems. The importance of these problems lies in the fact that they generates set of orthogonal functions. The sets of orthogonal functions are useful in the expansion of a certain class of functions.

S L equation : A classical "strum-lioulle equation", is a real second-order linear differential equation of the form

$$\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right]+q(x)y=\lambda r(x)y \qquad \ldots\ldots(1)$$

In the simplest of cases all coefficients are continuous on the finite closed interval $[a,b]$, and $P(x)$ has continuous derivatives. In this case y is called a "solution" if it is continuously differentiable on (a,b) and satisfies the equation (1) at every point in (a,b). In addition, the unknown function y is

required to satisfy boundary conditions. The function $r(x)$, is called the "weight" or "density" function. The number of famous differential equations could be represented in the SL form :

Bessel's equation : $x^2y''+xy'+(x^2-v^2)y=0$ can be written in strum-liouville form as

$$(xy')'+\left(x-\frac{v^2}{x}\right)y=0$$

The Legendre equation : $(1-x^2)y''-2xy'+v(v+1)y=0$ can easily be put into SL form, since $(1-x^2)'=-2x,$ so the Legendre equation is equivalent to $\left[(1-x^2)y'\right]'+v(v+1)y=0$

The general way to convert the 2nd order linear ODE to the SL form is to use an integrating factor $u(x)$ such that the $P(x)y''+Q(x)y'+R(x)y=0$ multiplied by $u(x)$ would has SL form, one can easily show that $u(x)=\frac{1}{P(x)}\exp\left(\int\frac{Q(x)}{P(x)}dx\right)$ does the job.

SL boundary value problem (SL-BVP) : We introduce the SL-operator as

$$L[y]=\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right]+q(x)y \quad \text{and consider the SL equation } L[y]+\lambda r(x)y=0 \qquad \ldots\ldots(2)$$

where $P(x)>0,\ r(x)>0$ and P,q and r are continuous functions on the interval $[a,b];$ along with the BC

$$\left.\begin{aligned}\alpha_1y(a)+\alpha_2P(a)y'(a)=0\\ \beta_1y(b)+\beta_2P(b)y'(b)=0\end{aligned}\right] \qquad \ldots\ldots(3)$$

where $\alpha_1^2+\alpha_2^2\neq 0$ and $\beta_1^2+\beta_2^2\neq 0$

The problem of finding a number λ such that BVP (2) – (3) has a non-trivial solution is called SLP. The value λ is called eigen value and the corresponding solution $(y:\lambda)$ is called an eigen function.

There are three types of SLP :

1. A SLP is called regular if $P(x)>0$ and $r(x)>0$ on $[a,b]$
2. A SLP is called singular if $P(x)>0$ on (a,b), $r(x)\geq 0$ on $[a,b]$ and $P(a)=P(b)=0$
3. A SLP is called periodic if $P(x)>0$, $r(x)>0$ and $P(x), q(x)$ and $r(x)$ are continuous functions on $[a,b]$; along with the following BC : $y(a)=y(b),\ y'(a)=y'(b)$

The most common types of SLP are regular and periodic.

Theorem (Orthogonality of characteristic function) :

Consider the Sturm Liouville problem consisting of the differential equation

$$\frac{d}{dt}\left[p(t)\frac{du}{dt}\right]+\left[q(t)+\lambda\ r(t)\right]u=0 \qquad \ldots\ldots.(1)$$

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where p , q , r are real functions s.*t.* $p(t)$ has a continuous derivative , $q(t)$ and $r(t)$ are continuous , and $p(t) > 0$ and $r(t) > 0$ for all t on a real interval $a \le t \le b$, and λ is a parameter independent of t , and the conditions

$$\left.\begin{aligned} A_1 u(a) + A_2 u'(a) &= 0 \\ B_1 u(b) + B_2 u'(b) &= 0 \end{aligned}\right\} \qquad \text{.......(2)}$$

where A_1 , A_2 , B_1 , B_2 are real constants s.*t.* A_1 , A_2 are not both zero and B_1 and B_2 are not both zero.

Let λ_m and λ_n be any two characteristic values of the problem. Let ϕ_m be a characteristic function corresponding to λ_m and let ϕ_n be a characteristic function corresponding to λ_n. Then characteristic functions ϕ_m and ϕ_n are orthogonal w.*r.t.* the weight function $r(t)$ on the interval $a \le t \le b$ i.e. $\int_a^b \phi_n(t)\, \phi_m(t)\, r(t) = 0$

Remark : Characteristic functions need not be orthonormal.

Common properties of regular/periodic BVP :

1. Eigen value are always real numbers.
2. Eigen vectors corresponding to distinct eigen values are L.I.
3. The eigen functions of a regular SLP corresponding to the distinct eigenvalues are orthogonal w.r.t. the weight function $r(x)$ on $[a,b]$. By otherwords, if the eigen functions u and v correspond to the distinct eigen values λ and μ then

$$\int_a^b r(x)u(x)v(x)\,dx = 0$$

4. If $A = \{\lambda \in \mathbb{R} : \text{differential equation has non-trivial solution}\}$ i.e. A is the set of eigen values. Then A is always infinite set. Also elements of A can be arranged into strictly increasing sequence $\lambda_1 < \lambda_2 < \lambda_3 < .... < \lambda_n < ....$ such that $\langle \lambda_n \rangle \to \infty$ as $n \to \infty$

$\Rightarrow$ A is always bounded below and unbounded above set. Again, every element of A is an isolated point.

$\Rightarrow$ A is no where dense set. Since set of isolated points is countable. So A is always countably infinite set.

5. There does not exist any $\lambda_0 \in \mathbb{R}$ such that the differential equation has non-trivial solution for every $\lambda > \lambda_0$.
6. There does not exist any $\lambda_0 \in \mathbb{R}$ such that differential equation has only trivial solution for all $\lambda > \lambda_0$.
7. There exists a $\lambda_0 \in \mathbb{R}$ such that the differential equation has only trivial solution for all $\lambda < \lambda_0$.
8. The number of negative eigen values are always finite if exists.

Properties of regular BVP :

1. The set A of eigen values is of the form $\lambda_1 < \lambda_2 < \lambda_3 < .... < \lambda_n < ....$ and $\langle \lambda_n \rangle \to +\infty$ as $n \to \infty$
2. The eigen values of the regular BVP are simple. Thus an eigen function that corresponds to an eigen value is unique upto a constant multiple $\Rightarrow$ $\dim[E_\lambda] = 1$
3. If $y_n(x)$ is an eigen vector or eigen function corresponding to λ_n, then $y_n(x)$ has exactly $(n-1)$ zeros in (a,b) $\Rightarrow$ corresponding of first eigen value, eigen function does not have any zeros and corresponding to λ_2 every eigen function has one zero and corresponding to λ_3 every eigen function has two zeros and so on.
4. Eigen function corresponding to least eigen value does not change its sign in $[a,b]$ or (a,b).
5. If $y(x)$ is an eigen function corresponding to λ_k, $k > 1$. Then $y(x)$ change its sign in $[a,b]$ or (a,b).
6. If $\lambda = 0$ is an eigen value and corresponding eigen function does not change its sign. This implies there does not exists any negative eigen value.
7. Similarly if $\lambda = 0$ is an eigen value and corresponding eigen function change its sign, then there must exists negative eigen values.

Properties of periodic BVP : Let $A = \{\lambda \in \mathbb{R} : \text{differential equation has non-trivial solution}\}$

$\lambda_1 < \lambda_2 < \lambda_3 < .... < \lambda_n < ....$

1. $\dim\left(E_{\lambda_0}\right) = 1$
2. $\dim\left(E_{\lambda_k}\right) = 2,$ for all $k \geq 1$
3. If $\lambda = 0$ is an eigen value and $\dim\left(E_0\right) = 1$, then there does not exists any negative eigen value.
4. If $\lambda = 0$ is an eigen value and $\dim\left(E_0\right) = 2$ then there must exists negative eigen value.

Example 1 : Show that the boundary value problem

$$\frac{d^2u}{dt^2} + \lambda u = 0 \qquad ...... (1)$$

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with conditions $u(0) = 0$, $u(\pi) = 0$ is a Sturm Liouville problem.

Solution : Given boundary value problem (1) can be written in the form $\frac{d}{dt}\left[1.\frac{du}{dt}\right]+\left[0+\lambda.1\right]u = 0$

and hence (1) is of the form $\frac{d}{dt}\left[p(t)\frac{du}{dt}\right]+\left[q(t)+\lambda\, r(t)\right]u = 0$

where $p(t) = 1$, $q(t) = 0$ and $r(t) = 1$. The supplementary conditions are of the special form

$$u(a) = 0 \quad , \quad u(b) = 0.$$

Example 2 : Show that the boundary value problem $\frac{d}{dt}\left[t\,\frac{du}{dt}\right]+\left[2t^2+\lambda\, t^3\right]u = 0$ ......(1)

with conditions $3\,u(1) + 4u'(1) = 0$ ..... (2)

$$5u(2) - 3\,u'(2) = 0$$

is a Sturm – Liouville problem.

Solution : The differential equation is of the form $\frac{d}{dt}\left[p(t)\frac{du}{dt}\right]+\left[q(t)+\lambda\, r(t)\right]u = 0$

where $p(t) = t$, $q(t) = 2t^2$ and $r(t) = t^3$. The conditions (2) are of the form

$$\alpha_1 u(a) + \alpha_2 u'(a) = 0$$
$$\beta_1 u(b) + \beta_2 u'(b) = 0$$

where $a = 1$, $b = 2$, $\alpha_1 = 3$, $\alpha_2 = 4$, $\beta_1 = 5$, $\beta_2 = -3$.

Example 3 : Find non-trivial solutions of Sturm Liouville boundary value problem

$$\frac{d^2u}{dt^2}+\lambda u = 0 \qquad ......(1)$$

$$u(0) = 0 \ , \ u(\pi) = 0 \qquad ......(2)$$

Solution : We shall consider separately the three cases $\lambda = 0$, $\lambda < 0$ and $\lambda > 0$. In each case , we shall first find the general solution of the differential equation (1) and then attempt to determine two arbitrary constants in this general solution so that the supplementary conditions (2) will also be satisfied.

Case(I) : $\lambda=0$ In this case (1) reduces to $\frac{d^2u}{dt^2} = 0$ and so the general solution is

$$u(t) = c_1 + c_2 t \qquad ....(3)$$

We now apply conditions (2) to solution (3). Condition $u(0) = 0$ implies

$$0 = c_1 + c_2.0 \quad \Rightarrow \quad c_1 = 0.$$

and condition $u(\pi) = 0$ implies $0 = c_1 + c_2\pi \quad \Rightarrow \quad c_2 = 0 \quad [\text{Since } c_1 = 0]$.

Thus in order that solution (3) to satisfy conditions (2) , we must have $c_1 = c_2 = 0$.

But then the solution (3) becomes $u(t) = 0$ for all t. Thus , in case when the parameter $\lambda = 0$, the only solution of the given problem is the trivial solution.

Case(II) : $\lambda<0$ Differential equation (1) is $\frac{d^2u}{dt^2}+\lambda u = 0$

Its auxiliary equation is $m^2+\lambda = 0$ and $m = \pm\sqrt{-\lambda}$. Since λ is negative , so these roots are real and unequal. Let us denote $\sqrt{-\lambda} = \alpha$, we have the general solution

$$u = c_1 e^{\alpha t} + c_2 e^{-\alpha t} \qquad ....(4)$$

We now apply boundary conditions (2) to the equation (4).

Condition $u(0) = 0$ implies $c_1 + c_2 = 0$ ....(5)

Condition $u(0) = \pi$ implies $c_1 e^{\alpha\pi} + c_2 e^{-\alpha\pi} = 0$ ....(6)

Clearly $c_1 = c_2 = 0$ is a solution of (5) and (6) , but these values of c_1 and c_2 would not give the non – trivial solution of the given problem. We must therefore seek non-zero values of c_1 and c_2 which satisfy (5) and (6).

This system has non-zero solution only if the determinant of coefficients is zero. Therefore , we must have

$$\begin{vmatrix} 1 & 1 \\ e^{\alpha\pi} & e^{-\alpha\pi} \end{vmatrix} = 0$$

$$\Rightarrow \quad e^{-\alpha\pi}-e^{\alpha\pi} = 0 \quad \Rightarrow \quad e^{-\alpha\pi} = e^{\alpha\pi} \quad \Rightarrow \quad e^{2\alpha\pi} = 1 \quad \Rightarrow \quad \alpha = 0.$$

Since $\alpha = \sqrt{-\lambda}$, we must have $\lambda = 0$. But $\lambda < 0$ in this case. Thus there are no non-trivial solution of the given problem in this case $\lambda < 0$.

Case(III) : $\lambda>0$ In this case *A.E.* is $m^2+\lambda = 0$. So its roots are $m = \pm\sqrt{-\lambda}$

These roots are conjugate complex numbers since $\lambda > 0$. Roots can be written as $\pm\sqrt{\lambda}\, i$. Thus in this case the general solution is of the form

$$u(t) = c_1\sin\sqrt{\lambda}\, t + c_2\cos\sqrt{\lambda}\, t \qquad ....(7)$$

We now apply conditions (2) to this general solution. Condition $u(0) = 0$ implies

$$c_1\sin 0 + c_2\cos 0 = 0 \quad \Rightarrow \quad c_2 = 0$$

Condition $u(\pi) = 0$ implies

$$c_1\sin\sqrt{\lambda}\,\pi + c_2\cos\sqrt{\lambda}\,\pi = 0$$

$$\Rightarrow \quad c_1\sin\sqrt{\lambda}\,\pi = 0 \qquad ....(8)$$

[Since $c_2 = 0$].

We must therefore satisfy (8).

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So we can either set $c_1 = 0$ or $\sin\sqrt{\lambda}\,\pi = 0$. However if $c_1 = 0$ then (since $c_2 = 0$ also) the solution (7) reduces to the trivial solution. Thus to obtain non-trivial solution we can not set $c_1 = 0$ but rather we must set

$\sin\sqrt{\lambda}\,\pi = 0 \Rightarrow \sqrt{\lambda} = n$, $n = 1, 2, 3\ldots\ldots$ [Since $\sqrt{\lambda}$ is positive]

$\Rightarrow \lambda = n^2$, $n = 1, 2, 3\ldots\ldots$

Therefore in order that the differential equation (1) have a non-trivial solution of the form (7) satisfying the condition (2) , we must have

$\lambda = n^2$ where $n = 1, 2, 3\ldots\ldots$

In other words , the parameter λ in (1) must be a number of the infinite sequence 1 , 4 , 9 , 16………
Also from (7) we see that non-trivial solutions corresponding to $\lambda = n^2$ ($n = 1, 2, 3\ldots\ldots$) are given by $u(t) = c_n \sin nt$, where c_n is arbitrary non-zero constant.

Def. Characteristic Values and Characteristic functions :

Consider the Sturm – Liouville problem consisting of the differential equation

$$\frac{d}{dt}\left[p(t)\frac{du}{dt}\right] + [q(t) + \lambda\, r(t)]u = 0 \qquad \ldots\ldots(1)$$

and the supplementary conditions

$$\left.\begin{aligned} \alpha_1 u(a) + \alpha_2 u'(a) &= 0 \\ \beta_1 u(b) + \beta_2 u'(b) &= 0 \end{aligned}\right\} \qquad \ldots\ldots(2)$$

The values of parameter λ in (1) for which these exist non-trivial solutions of the Sturm Liouville problem are called the characteristic values (or eigen values of the problem). The corresponding non-trivial solutions are called the characteristic functions or the eigen functions of the problem.

Example 4 : Find the characteristic functions of the strum-liouville problem

$y'' + \lambda y = 0,\ y(0) - y(\pi) = 0,\ y'(0) - y'(\pi) = 0$, $A \in \mathbb{R}$

Solution : $y'' + \lambda y = 0,\ y(0) - y(\pi) = 0,\ y'(0) - y'(\pi) = 0$

These boundary conditions are called periodic boundary condition. We consider three cases corresponding to the value of λ :

Case I : $\lambda = -u^2 < 0$

The general solution of the ODE is given as $yy = Ae^{-ux} + Be^{-ux}$

By substituting BC we obtain the following system :

$$\begin{cases} A(1-e^{-u\pi})+B(1-e^{u\pi})=0 \\ A(-1+e^{-u\pi})+B(1-e^{u\pi})=0 \end{cases}$$

This system has only trivial solution $A=B=0$ (for $u \neq 0$)

Case II : $\lambda = 0$

In this case the problem has a solution $y = Ax + B$ and by substituting BC we obtain $A = 0$ and B is an arbitrary constants. This corresponds to the eigen value $\lambda_0 = 0$ and the eigen function $\phi_0 = 1$ (we set $B = 1$).

Note that this eigen value is simple. The eigen value is called simple, if its eigenspace is of dimension one ; otherwise the eigenvalue is called multiple.

Case III : $\lambda = u^2 > 0$ The general solution of the ODE is given as $y = A\cos(ux) + B\sin(ux)$

By substituting BC we obtain the following system :

$$\begin{bmatrix} A(1-\cos(u\pi)-B\sin(u\pi)=0 \\ A\sin(u\pi)+B\sin(1-\cos(u\pi))=0 \end{bmatrix}$$

This problem has a non-trivial solution only when the determinant of the matrix of coefficients $D(u) = 2 - 2\cos\pi = 0$. This corresponds to $u = 2n,\ \ n = \pm 1, \pm 2, ...$ and hence $\lambda_n = 4n^2$.

The eigen functions corresponding to λ_n are given by $(A = B = 1)$

$$\phi_n = \cos\left(\sqrt{\lambda_n}x\right),\ \psi_n = \sin\left(\sqrt{\lambda_n}\,x\right)$$

Note that the eigen values λ_n are positive and there are two linearly independent eigen functions corresponding to each eigen value, so they are not unique.

Example 5 : Find the characteristic values and characteristic functions of the Sturm – Liouville problem

$$\frac{d}{dx}\left[x\ \frac{dy}{dx}\right] + \frac{\lambda}{x}y \ = \ 0 \qquad ....(1)$$

$$y'(1) \ = \ 0 \quad , \quad y'\left(e^{2\pi}\right) \ = \ 0 \qquad ....(2)$$

where we assume that the parameter λ is non-negative.

Solution : Differential equation (1) can be written as $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + \dfrac{\lambda}{x}y \ = \ 0$

or

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \lambda y \ = \ 0 \qquad ....(3)$$

Putting x = e^t in (3) we get

$$\frac{dy}{dx} \ = \ \frac{dy}{dt}\cdot\frac{dt}{dx} \ = \ \frac{1}{x}\cdot\frac{dy}{dt} \quad \Rightarrow \quad x\frac{dy}{dx} \ = \ \frac{dy}{dt} \quad \text{and} \quad x^2\frac{d^2y}{dx^2} \ = \ \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

Using these values in (3) , we get

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$$\frac{d^2y}{dt^2}+\lambda y = 0 \qquad ....(4)$$

We consider the separately two cases.

Case (i) : $\lambda=0$ Differential equation (4) reduces to $\frac{d^2y}{dt^2} = 0$

The general solution of this differential equation is $y = c_0 t + c$

or $\quad y = c_0 \log x + c \qquad ....(5)$

Now we apply the conditions (2) to solution (5). Condition $y'(1) = 0$ implies

$$0 = \frac{c_0}{1} \Rightarrow c_0 = 0 \quad [\text{Since } y'(x) = \frac{c_0}{x}]$$

Condition $y'\left(e^{2\pi}\right) = 0$ implies , $c_0 = 0$

Both of these conditions does not impose any restriction upon c. Thus for $\lambda = 0$, we obtain the solutions $y = c$ where c is an arbitrary constant.

These are non-trivial solutions for all $c \neq 0$. Thus $\lambda = 0$ is a characteristic value and the corresponding characteristic functions are given by $y = c$, where c is an arbitrary non-zero constant.

Case (ii) : $\lambda>0$ In this case $A.E.$ is $m^2+\lambda = 0 \Rightarrow m = \pm\sqrt{-\lambda} = \pm\sqrt{\lambda}i$.

Thus the general solution of (4) may be written as $y = c_1\sin\sqrt{\lambda}\, t + c_2\cos\sqrt{\lambda}\, t$.

Thus the general solution of (1) is

$$y = c_1\sin(\sqrt{\lambda} \log x) + c_2\cos(\sqrt{\lambda} \log x) \qquad [\text{Since } t = \log x] \qquad ....(6)$$

We now apply the boundary conditions (2) to a solution (6). From (6) , we get

$$\frac{dy}{dx} = y'(x) = \frac{c_1\sqrt{\lambda}}{x}\cos(\sqrt{\lambda} \log x) - \frac{c_2\sqrt{\lambda}}{x}\sin(\sqrt{\lambda} \log x) \qquad ....(7)$$

Condition $y'(1) = 0$ implies $c_1\sqrt{\lambda}\cos(\sqrt{\lambda} \log 1) - c_2\sin(\sqrt{\lambda} \log 1) = 0$ or $c_1\sqrt{\lambda} = 0$ [Since log 1= 0]

Thus we must have $c_1 = 0 \qquad ....(8)$

Condition $y'\left(e^{2\pi}\right) = 0$ implies

$$c_1\sqrt{\lambda}\, e^{-2\pi}\cos\left(\sqrt{\lambda}\, \log e^{2\pi}\right) - c_2\sqrt{\lambda}\, e^{-2\pi}\sin\left(\sqrt{\lambda}\, \log e^{2\pi}\right) = 0$$

$$\Rightarrow \quad c_2\sqrt{\lambda}\, e^{-2\pi}\sin\left(2\pi\sqrt{\lambda}\right) = 0 \quad [c_1 = 0]. \text{ Either } c_2 = 0 \text{ or } \sin\left(2\pi\sqrt{\lambda}\right) = 0$$

Since $c_1 = 0$, the choice $c_2 = 0$ would give us the trivial solution. So we must have $\sin\left(2\pi\sqrt{\lambda}\right) = 0$ and hence

$$2\pi\sqrt{\lambda} = n\pi \ , \ n = 1, 2, \ldots\ldots\ldots$$

Thus for the non-trivial solution , we must have $\lambda = \dfrac{n^2}{4}$, $n = 1, 2, 3, \ldots\ldots\ldots$ ……..(9)

Corresponding to these values of λ , we obtain the non-trivial solutions

$$y = c_n \cos\left(\frac{n \log x}{2}\right) \qquad (n = 1, 2, 3, \ldots\ldots\ldots) \qquad \ldots\ldots(10)$$

where c_n 's are arbitrary non-zero constants.

Analysis of solutions of linear differential equation :

$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$ ……(1)

$a_0(x) \neq 0 \quad \forall\ x$, $a_0(x), a_1(x), a_2(x)$ are continuous for all x.

Properties of Non-trivial solution of (1) is

1. If $y(x)$ be non-zero solution then zeros of $y(x)$ are simple zeros. i.e., if $y(\alpha) = 0$ then $y'(\alpha) \neq 0$.

2. If $A = \{\alpha \in R : y(\alpha) = 0\}$, where $y(x)$ is non zero solution. Then

(i) A is always countable set

(ii) $d(A) = A' = \phi$, i.e., A is nowhere dense.

(iii) every element of A is isolated point.

(iv) A must be closed set ($\because$ every point is isolated and A' is empty).

Property of two linearly independent solution of linear differential equation :

Let $y_1(x)$ and $y_2(x)$ are two linearly independent solution and

$$A = \{\alpha \in R : y_1(\alpha) = 0\},$$

$$B = \{\beta \in R : y_2(\beta) = 0\}$$

Then

(i) A and B must be countable.

(ii) $A \cap B = \phi$, i.e., zeros of two L.I. solution never common zero or if two solution have common zero they must be L.D.

(iii) Between any two successive zeros of $y_1(x), \exists$ exactly one zero of $y_2(x)$ and conversely.

(iv) If cardinality of A is finite iff card B is finite.

(v) If $\text{card}(A)$ is finite $\Rightarrow$ $|\text{card}\, A - \text{card}\, B| \in \{0,1\}$

Let us consider another type of differential equation :

$y'' + q(x)y = 0$

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Case I . $q(x)$ is continuous and $q(x) < 0$ for all x.

Let $y(x)$ be a non trivial solution then

(i) $y(x)$ can have atmost one zero on R.

(ii) If $y(x)$ has zero in R then $y'(x)$ never zero.

(iii) If $y'(x)$ has a zero in R $y(x)$ never zero.

(iv) $y(x)$ is monotonic function and $y'(x)$ is also monotonic function.

(v) $\lim_{x\to\infty} y(x) \to \infty$ and $\lim_{x\to\infty} y'(x) \to \infty$.

$\mathbf{y''(x)+q(x)y=0}$

Case II. **$q(x)$ is continuous $q(x)>0$ $\forall$ $x \in R^+$ assuming that $\int_1^\infty q(x)dx = \infty$**

If $y(x)$ be a non trivial solution then

(i) $y(x)$ has countably infinite zeros on R^+.

(ii) $y'(x)$ has countably infinite zeros on R^+.

(iii) $y(x)$ and $y'(x)$ are non monotonic function on R^+.

Particular case :

$q(x)$ is continuous on a bounded interval $[a,b]$, so $q(x)$ is bounded.

So $\quad 0 < m < q(x) < M$ for all x,

m and M are infimum and supremum respectively.

(iv) If $y(x)$ be a non trivial solution and if x_0, x_1 are successive zeros of $y(x)$, let $x_0 < x_1$ then

$$\frac{\pi}{\sqrt{M}} < x_1 - x_0 < \frac{\pi}{\sqrt{m}}$$

(i) Distance between two successive zeros of any non-trivial solution are lying between $\left(\frac{\pi}{\sqrt{M}}, \frac{\pi}{\sqrt{m}}\right)$

(ii) If $m \geq \frac{K^2\pi^2}{(b-a)^2}$ then $y(x)$ has atleast K zeros in $[a,b]$.

Exercise 5.4

Find the characteristic values and characteristic functions of each of the following Sturm Liouville problems.

1. $\frac{d^2y}{dx^2}+\lambda y=0$; $y(0) = 0,\ y(\pi)=0.$

2. $\frac{d^2y}{dx^2}+\lambda y=0$; $y(0) = 0,\ y\left(\frac{\pi}{2}\right)=0.$

3. $\frac{d^2y}{dx^2}+\lambda y=0$; $y(0) = 0,\ y(2\pi)=0.$

4. $\frac{d^2y}{dx^2}+\lambda y=0$; $y(0) = 0,\ y(l)=0.$

5. $\frac{d^2y}{dx^2}+\lambda y=0$; $y(0) = 0,\ y'(\pi)=0.$

6. $\frac{d^2y}{dx^2}+\lambda y=0$; $y(0) = 0,\ y'\left(\frac{\pi}{2}\right)=0.$

7. $\frac{d^2y}{dx^2}+\lambda y=0$; $y(0) = 0,\ y'(2\pi)=0.$

8. $\frac{d^2y}{dx^2}+\lambda y=0$; $y(0) = 0,\ y'(l)=0.$

9. $\frac{d^2y}{dx^2}+\lambda y=0$; $y'(0) = 0,\ y(\pi)=0.$

10. $\frac{d^2y}{dx^2}+\lambda y=0$; $y'(0) = 0,\ y\left(\frac{\pi}{2}\right)=0.$

11. $\frac{d^2y}{dx^2}+\lambda y=0$; $y'(0) = 0,\ y(2\pi)=0.$

12. $\frac{d^2y}{dx^2}+\lambda y=0$; $y'(0) = 0,\ y(l)=0.$

13. $\frac{d^2y}{dx^2}+\lambda y=0$; $y'(0) = 0,\ y'(\pi)=0.$

14. $\frac{d^2y}{dx^2}+\lambda y=0$; $y'(0) = 0,\ y'\left(\frac{\pi}{2}\right)=0.$

15. $\frac{d^2y}{dx^2}+\lambda y=0$; $y'(0) = 0,\ y'(2\pi)=0.$

16. $\frac{d^2y}{dx^2}+\lambda y=0$; $y'(0) = 0,\ y'(l)=0.$

17. $\frac{d^2y}{dx^2}+\lambda y=0$; $y'(-\pi) = 0,\ y'(\pi)=0.$

18. $\frac{d}{dx}\left[x\frac{dy}{dx}\right]+\frac{\lambda}{x}y=0,$ $y(1) = 0, y\left(e^{\pi}\right)=0.$

19. $\frac{d}{dx}\left[x\frac{dy}{dx}\right]+\frac{\lambda}{x}y=0;$ $y(1) = 0,\ y'\left(e^{\pi}\right)=0.$

20. $\frac{d^2y}{dx^2}+\lambda y=0, y(0) = 0,\ y(\pi)-y'(\pi)=0.$

21. $\frac{d^2y}{dx^2}+\lambda y=0,\ y(0)-y'(0)=0,\ y(\pi)-y'(\pi)=0.$

22. $\frac{d^2y}{dx^2}+\lambda y=0,\ y(0)+y'(0)=0,\ y(1)+y'(1)=0.$

23. $\frac{d^2y}{dx^2}+\lambda y=0,\ y(0)=y(2c),\ y'(0)=y'(2c).$

24. $\frac{d}{dx}\left[x\frac{dy}{dx}\right]+\frac{\lambda}{x}y=0,\quad y'(1)=y'\left(e^{2\pi}\right)=0$

25. $\frac{d}{dx}\left[x^3\frac{dy}{dx}\right]+\lambda xy=0$; $y(1) = 0, y(e) = 0.$

26. $y''+2y'+(1+\lambda)y=0$; $y(0) = 0, y'(l)=0.$

27. $\frac{d}{dx}\left[\left(x^2+1\right)\frac{dy}{dx}\right]+\frac{\lambda}{x^2+1}y=0, y(0) = 0, y(1) = 0.$ [Hint : Let $x = \tan t$]

28. $\frac{d}{dx}\left[\left(\frac{1}{3x^2+1}\right)\frac{dy}{dx}\right]+\lambda\left(3x^2+1\right)y=0,$ $y(0) = 0, y(\pi)=0.$ [Hint : Let $t=x^3+x$]

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Answers

1. $\lambda = n^2, \quad n = 1,2,3,.....$ $\qquad y(x) = c\sin nx, \ n = 1,2,3,.....$

2. $\lambda = 4n^2, \quad n = 1,2,3,.....$ $\qquad y(x) = c\sin 2nx, \ n = 1,2,3,.....$

3. $\lambda = \dfrac{n^2}{4}, \quad n = 1,2,3,.....$ $\qquad y(x) = c\sin\dfrac{nx}{2}, \ n = 1,2,3,.....$

4. $\lambda = \dfrac{n^2\pi^2}{l^2}, \quad n = 1,2,3,.....$ $\qquad y(x) = c\sin\left(\dfrac{n\pi x}{l}\right), \ n = 1,2,3,.....$

5. $\lambda = \left(n+\dfrac{1}{2}\right)^2, \quad n = 0,1,2,3,.....$ $\qquad y(x) = c\sin\left(n+\dfrac{1}{2}\right)x, \ n = 0,1,2,3,.....$

6. $\lambda = (2n+1)^2, \quad n = 0,1,2,.....$ $\qquad y(x) = c\sin(2n+1)x, \ n = 0,1,2,3,.....$

7. $\lambda = \left(\dfrac{2n+1}{4}\right)^2, \quad n = 0,1,2,3,.....$ $\qquad y(x) = c\sin\left(\dfrac{2n+1}{4}\right)x, \ n = 0,1,2,3,.....$

8. $\lambda = (2n+1)^2\dfrac{\pi^2}{4l^2}, \quad n = 0,1,2,3,.....$ $\qquad y(x) = c\sin\left((2n+1)\dfrac{\pi}{2l}\right)x, \ n = 0,1,2,3,.....$

9. $\lambda = \left(n+\dfrac{1}{2}\right)^2, \quad n = 0,1,2,3,.....$ $\qquad y(x) = c\cos\left(n+\dfrac{1}{2}\right)x, \ n = 0,1,2,3,.....$

10. $\lambda = (2n+1)^2, \quad n = 0,1,2,3,.....$ $\qquad y(x) = c\cos(2n+1)x, \ n = 0,1,2,3,.....$

11. $\lambda = \left(\dfrac{2n+1}{4}\right)^2, \quad n = 0,1,2,3,.....$ $\qquad y(x) = c\cos\left(\dfrac{2n+1}{4}\right)x, \ n = 0,1,2,3,.....$

12. $\lambda = (2n+1)^2\dfrac{\pi^2}{4l^2}, \quad n = 0,1,2,3,.....$ $\qquad y(x) = c\cos\left((2n+1)\dfrac{\pi}{2l}\right)x, \ n = 0,1,2,3,.....$

13. $\lambda = n^2, \quad n = 0,1,2,3,.....$ $\qquad y(x) = c\cos(nx), \ n = 0,1,2,3,.....$

14. $\lambda = 4n^2, \quad n = 0,1,2,3,.....$ $\qquad y(x) = c\cos(2nx), \ n = 0,1,2,3,.....$

15. $\lambda = \dfrac{n^2}{4}, \quad n = 0,1,2,3,.....$ $\qquad y(x) = c\cos\dfrac{nx}{2}, \ n = 0,1,2,3,.....$

16. $\lambda = \dfrac{n^2\pi^2}{l^2}, \quad n = 0,1,2,3,.....$ $\qquad y(x) = c\cos\left(\dfrac{n\pi x}{l}\right), \ n = 0,1,2,3,.....$

17. $\lambda = n^2, \quad n = 1, 2, 3, ...$ $\qquad$ $y(x) = c\cos(nx), \; n = 1, 2, 3, ...$

$\lambda = \left(n + \frac{1}{2}\right)^2, \quad n = 0, 1, 2, 3, ...$ $\qquad$ $y(x) = c\sin\left(n + \frac{1}{2}\right)x, \; n = 0, 1, 2, 3, ...$

18. $\lambda = n^2, \quad n = 1, 2, 3, ...$ $\qquad$ $y(x) = c\sin(n\log|x|), \; n = 1, 2, 3, ...$

19. $\lambda = \left(n + \frac{1}{2}\right)^2, \quad n = 0, 1, 2, 3, ...$ $\qquad$ $y(x) = c\sin\left[\left(n + \frac{1}{2}\right)\log|x|\right], \; n = 0, 1, 2, 3, ...$

20. $\lambda = k^2$ where $k = \tan k\pi$ $\qquad$ $y(x) = c\sin kx$

21. $\lambda = -1$ $\qquad$ $y(x) = ce^x$

$\lambda = n^2, \; n = 1, 2, 3, ...$ $\qquad$ $y(x) = c(n\cos nx + \sin nx); n = 1, 2, 3, ...$

22. $\lambda = -1$ $\qquad$ $y(x) = ce^{-x}$

$\lambda = (n\pi)^2, \; n = 1, 2, 3, ...$ $\qquad$ $y(x) = c(-n\pi\cos n\pi x + \sin n\pi x); n = 1, 2, 3, ...$

23. $\lambda = \left(\frac{n\pi}{c}\right)^2, \quad n = 0, 1, 2, 3, ...$ $\qquad$ $y(x) = A\cos\left(\frac{n\pi}{c}\right)x + B\sin\left(\frac{n\pi}{c}\right)x \; ; \; n = 0, 1, 2, ...$

24. $\lambda = \left(\frac{n}{2}\right)^2, \quad n = 0, 1, 2, 3, ...$ $\qquad$ $y(x) = c\cos\left(\frac{n}{2}\log|x|\right); \; n = 0, 1, 2, ...$

25. $\lambda = 1 + n^2\pi^2, \; n = 1, 2, 3, ...$ $\qquad$ $y(x) = \frac{c}{x}\sin\left(n\pi\log|x|\right); \; n = 1, 2, ...$

26. $\lambda = k^2$, where k is a non-zero solution of $\tan kl = k$ $\qquad$ $y(x) = ce^{-x}\sin kx$

27. $\lambda = (4n)^2 \; ; \; n = 1, 2, ...$ $\qquad$ $y(x) = c\sin(4n\tan^{-1}x) \; ; \; n = 1, 2, 3, ...$

28. $\lambda = \left(\frac{n}{\pi^2 + 1}\right)^2 \; ; \; n = 1, 2, 3, ...$ $\qquad$ $y(x) = c\sin\left[\left(\frac{n}{\pi^2 + 1}\right)(x^3 + x)\right], \quad n = 1, 2, 3, ...$

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Assignment-5

--------------------- **S C Q** ---------------------

1. The differential equation whose linearly independent solutions are $\cos 2x$, $\sin 2x$ and e^{-x} is
   1. $\left(D^3+D^2+4D+4\right)y=0$
   2. $\left(D^3-D^2+4D-4\right)y=0$
   3. $\left(D^3+D^2-4D-4\right)y=0$
   4. $\left(D^3-D^2-4D+4\right)y=0$

2. For which of the following pair of functions $y_1(x)$ and $y_2(x)$, continuous function $P(x)$ and $q(x)$ can be determined on $[-1,1]$ such that $y_1(x)$ and $y_2(x)$ give two linearly independent solutions of $y''+P(x)y'+q(x)y=0,\ \ x\in[-1,1]$
   1. $y_1(x)=x\sin x,\ y_2(x)=\cos x$
   2. $y_1(x)=xe^x,\ y_2(x)=\sin x$
   3. $y_1(x)=e^{x-1},\ y_2(x)=e^x-1$
   4. $y_1(x)=x^2,\ y_2(x)=\cos x$

3. Let $y_1(x)=1+x$ and $y_2(x)=e^x$ be two solutions of $y''+P(x)y'+Q(x)y=0$, then $P(x)$ is equal to
   1. $1+x$
   2. $-1-x$
   3. $\dfrac{1+x}{x}$
   4. $\dfrac{-1-x}{x}$

4. Let $y_1(x)=1+x$ and $y_2(x)=e^x$ be two solutions of $y''+P(x)y'+Q(x)y=0$. The set of initial condition for which the above differential equation has no solution is
   1. $y(0)=2,\ \ y'(0)=1$
   2. $y(1)=0,\ \ y'(1)=1$
   3. $y(1)=1,\ \ y'(1)=0$
   4. $y(2)=1,\ \ y'(2)=2$

5. The set of linearly independent solutions of the differential equation $\dfrac{d^4y}{dx^4}-\dfrac{d^2y}{dx^2}=0$ is
   1. $\left\{1,x,e^x,e^{-x}\right\}$
   2. $\left\{1,x,e^{-x},xe^{-x}\right\}$
   3. $\left\{1,x,e^x,xe^x\right\}$
   4. $\left\{1,x,e^x,xe^{-x}\right\}$

6. The maximum number of linearly independent solutions of the differential equation $\dfrac{d^4y}{dx^4}=0$, which the condition $y(0)=1$, is
   1. 4
   2. 3
   3. 2
   4. 1

7. Let V be the set of all bounded solutions of the ODE $u''(t)-4u'(t)+3u(t)=0$, $t\in\mathbb{R}$. Then V
   1. is a real vector space of dimension 2.
   2. is a real vector space of dimension 1.
   3. contains only the trivial function $u=0$.
   4. contains exactly two functions.

8. Consider the ODE on $\mathbb{R}$, $y'(x)=f\left(y(x)\right)$. If f is an even function and y is an odd function, then

1. $-y(-x)$ is also a solution
2. $y(-x)$ is also a solution
3. $-y(x)$ is also a solution
4. $y(x)y(-x)$ is also a solution

9. Let $y=\phi(x)$ and $y=\psi(x)$ be solutions of $y''-2xy'+(\sin x^2)y=0$ such that $\phi(0)=1$, $\phi'(0)=1$ and $\psi(0)=1$, $\psi'(0)=2$. Then, the value of the wronskian $W(\phi,\psi)$ at $x=1$ is

1. 0   2. 1   3. e 4. e^2

10. Let y_1 and y_2 be two linearly independent solutions of $y''+(\sin x)y=0$, $0\le x\le 1$. Let $g(x)=W(y_1,y_2)(x)$ be wronskian of y_1 and y_2. Then

1. $g'>0$ on $[0,1]$
2. $g'<0$ on $[0,1]$
3. g' vanishes at only one point of $[0,1]$
4. g' vanishes at all points of $[0,1]$

11. Consider the ODE

$u''(t)+P(t)u'(t)+Q(t)u(t)=R(t),\ t\in[0,1]$

There exist continuous functions P,Q and R defined on $[0,1]$ and two solutions u_1 and u_2 of this ODE such that wronskian of u_1 and u_2 is

1. $W(t)=2t-1,\ 0\le t\le 1$
2. $W(t)=\sin 2\pi t,\ 0\le t\le 1$
3. $W(t)=\cos 2\pi t,\ 0\le t\le 1$
4. $W(t)=1,\ 0\le t\le 1$

12. Let $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+q(x)y=0,\ a\le x\le b.$$

Where $P(x)$ and $q(x)$ are real valued continuous functions on $[a,b]$. If x_0 and x_1, with $x_0<x_1$, are consecutive zeros of $y_1(x)$ in (a,b) then

1. $y_1(x)=(x-x_0)q_0(x)$, where $q_0(x)$ is continuous on $[a,b]$ with $q_0(x_0)\ne 0$
2. $y_1(x)=(x-x_0)^2P_0(x)$, where $P_0(x)$ is continuous on $[a,b]$ with $P_0(x_0)\ne 0$
3. $y_2(x)$ has no zero in (x_0,x_1)
4. $y_2(x_0)=0$ but $y_2'(x_0)\ne 0$

13. Let $Y_1(x)$ and $Y_2(x)$ defined on $[0,1]$ be twice continuously differentiable functions satisfying $Y''(x)+Y'(x)+Y(x)=0$. Let $W(x)$ be the wronskian of Y_1 and Y_2 and satisfy $W\left(\frac{1}{2}\right)=0$. Then

1. $W(x)=0$ for $x\in[0,1]$
2. $W(x)\ne 0$ for $x\in\left[0,\frac{1}{2}\right]\cup\left(\frac{1}{2},1\right]$
3. $W(x)>0$ for $x\in\left(\frac{1}{2},1\right]$
4. $W(x)<0$ for $x\in\left[0,\frac{1}{2}\right)$

14. Let y_1 and y_2 be two solutions of the problem $\left.\begin{array}{l} y''(t)+ay'(t)+by(t)=0,\ t\in\mathbb{R} \\ y(0)=0 \end{array}\right\}$

Where a and b are real constants. Let W be the wronskian of y_1 and y_2. Then

1. $W(t)=0,\ \forall\ t\in\mathbb{R}$

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2. $W(t)=c,\ \forall\ t\in\mathbb{R}$ for some positive constant c
3. W is a non-constant positive function
4. There exists $t_1, t_2\in\mathbb{R}$ such that $W(t_1)<0<W(t_2)$

15. The boundary value problem $x^2y''-2xy'+2y=0,$ subject to the boundary conditions $y(1)+\alpha y'(1)=1,$ $y(2)+\beta y'(2)=2$ has a unique solution if

1. $\alpha=-1,\quad \beta=2$
2. $\alpha=-1,\quad \beta=-2$
3. $\alpha=-2,\quad \beta=2$
4. $\alpha=-3,\quad \beta=\frac{2}{3}$

16. The strum Liouville problem $y''+\lambda^2y=0,\ y(0)=0,\ y'(\pi)=0$ has its eigen vectors given by y is equal to

1. $\sin\left(n+\frac{1}{2}\right)x$
2. $\sin nx$
3. $\cos\left(n+\frac{1}{2}\right)x$
4. $\cos nx,$ where $n=0,1,2,\ldots$

17. The eigen values of the strum Liouville system $y''+\lambda y=0,\ 0\le x\le\pi,$ $y(0)=0,\ y'(\pi)=0$ are

1. $\frac{n^2}{4}$
2. $\frac{(2n-1)^2\pi^2}{4}$
3. $\frac{(2n+1)^2}{4}$
4. $\frac{n^2\pi^2}{4}$

18. The set of all eigen values of the strum Liouville problem $y''+\lambda y=0,\ y'(0)=0,$ $y'\left(\frac{\pi}{2}\right)=0$ is given by

1. $\lambda=2n,\ n=1,2,3,\ldots$
2. $\lambda=2n,\ n=0,1,2,3,\ldots$
3. $\lambda=4n^2,\ n=1,2,3,\ldots$
4. $\lambda=4n^2,\ n=0,1,2,3,\ldots$

19. Let n be a non-negative integer. The eigen values of the sturm Liouville problem $\frac{d^2y}{dx^2}+\lambda y=0,$ with boundary conditions $y(0)=y(2\pi),\ \frac{dy}{dx}(0)=\frac{dy}{dx}(2\pi)$ are

1. n
2. $n^2\pi^2$
3. $n\pi$
4. n^2

20. The set of real numbers λ for which the boundary value problem $\frac{d^2y}{dx^2}+\lambda y=0,\ y(0)=0,\ y(\pi)=0$ has nontrivial solutions is

1. $(-\infty,0)$
2. $\{\sqrt{n}\mid n \text{ is a positive integer}\}$
3. $\{n^2\mid n \text{ is a positive integer}\}$
4. $\mathbb{R}$

(CSIR NET Dec 2017)

21. Consider the ordinary differential equation $y''+P(x)y'+Q(x)y=0$ where P and Q are smooth functions. Let y_1 and y_2 be any two solutions of the ODE. Let $W(x)$ be the

corresponding Wronskian. Then which of the following is always true ?

1. If y_1 and y_2 are linearly dependent then $\exists\ x_1, x_2$ such that $W(x_1) = 0$ and $W(x_2) \neq 0$
2. If y_1 and y_2 are linearly independent then $W(x) = 0 \ \ \forall\ x$
3. If y_1 and y_2 are linearly dependent then $W(x) \neq 0 \ \ \forall\ x$
4. If y_1 and y_2 are linearly independent then $W(x) \neq 0 \ \ \forall\ x$

(CSIR NET June 2018)

------------------------- **M C Q** --------------------

1. Let $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions to the differential equation $y'' + p(x)y' + q(x)y = 0$, $a \le x \le b$, where $p(x)$ and $q(x)$ are continuous in $[a,b]$, and x_0 is a point in (a,b). Then
   1. both $y_1(x)$ and $y_2(x)$ cannot have a local maximum at x_0.
   2. both $y_1(x)$ and $y_2(x)$ cannot have a local minimum at x_0
   3. $y_1(x)$ cannot have a local maximum at x_0 and $y_2(x)$ cannot have local minimum at x_0 simultaneously.
   4. both $y_1(x)$ and $y_2(x)$ cannot vanish at x_0 simultaneously.
2. For the boundary value problem $y'' + \lambda y = 0,\ y(-\pi) = y(\pi),\ y'(-\pi) = y'(\pi)$ to each non-zero eigen value λ. There correspond
   1. only one eigenfunction
   2. two eigen function
   3. two linearly independent eigen functions
   4. two orthogonal eigen functions
3. Let $\dfrac{d^2y}{dx^2} - q(x)y = 0,\ 0 \le x < \infty$ $y(0) = 1,\ \dfrac{dy}{dx}(0) = 1,$ where $q(x)$ is a positive monotonically increasing continuous function. Then
   1. $y(x) \to \infty$ as $x \to \infty$
   2. $\dfrac{dy}{dx} \to \infty$ as $x \to \infty$
   3. $y(x)$ has finitely many zeros in $[0,\infty)$
   4. $y(x)$ has infinitely many zeros in $[0,\infty)$
4. For the boundary value problem $y'' + \lambda y = 0$; $y(0) = 0,\ y(1) = 0$, there exists an eigen value λ for which there corresponds an eigen function in $(0,1)$ that
   1. does not change sign
   2. change sign
   3. is positive
   4. is negative
5. The solution of the boundary value problem $\dfrac{d^2y}{dx^2} + y = \operatorname{cosec} x$; $0 < x < \dfrac{\pi}{2}$, $y(0) = 0,\ y\left(\dfrac{\pi}{2}\right) = 0$ is
   1. convex
   2. concave
   3. negative
   4. positive

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6. Consider the boundary value problem

$-u''(x) = \pi^2 u(x)$; $x \in (0,1)$
$u(0) = u(1) = 0$

If u and u' are continuous on $[0,1]$; then

1. $u'^2(x) + \pi^2 u^2(x) = u'^2(0)$
2. $\int_0^1 u'^2(x)dx - \pi^2 \int_0^1 u^2(x)dx = 0$
3. $u'^2(x) + \pi^2 u^2(x) = 0$
4. $\int_0^1 u'^2(x)dx - \pi^2 \int_0^1 u^2(x)dx = u'^2(0)$

7. Consider the boundary value problem (BVP) $u'' + \lambda u = 0,\ u(0) = u'(\pi) = 0,$

$u'' = \dfrac{d^2u}{dx^2},\ \lambda \in \mathbb{C}$

Let K denote a nonnegative integer. Then which of the following are correct ?

1. There exist eigenvalues of the BVP and the corresponding eigen functions constitute an orthogonal set.
2. The eigenvalues of the BVP are $\left(K + \frac{1}{2}\right)^2$ with the corresponding eigenfunctions $\left\{\sin\left(K + \frac{1}{2}\right)x\right\}$
3. The eigenvalues of the BVP are $(K+1)^2$ with the corresponding eigenfunctions $\{\sin(K+1)x\}$
4. There exists no nonreal eigenvalues for the BVP.

8. Let y be a nontrivial solution of the boundary value problem $y'' + xy = 0$, $x \in [a,b]$, $y(a) = y(b) = 0$ then

1. $b > 0$
2. y is monotone in $(a,0)$ if $a < 0 < b$
3. $y'(a) = 0$
4. y has infinitely many zeros in $[a,b]$

9. Let $y_1(x)$ and $y_2(x)$ form a complete set of solutions of the differential equation

$y'' - 2xy' + \sin\left(e^{2x^2}\right)y = 0,\ x \in [0,1]$ with

$y_1(0) = 0,\ y'(0) = 1,\ y_2(0) = 1,\ y'_2(0) = 1.$

Then the wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ at $x = 1$ is

1. e^2
2. $-e$
3. $-e^2$
4. e

10. Let P, Q be continuous real valued functions defined on $[-1,1] \to R,\ i = 1,2$ be solutions of the ODE :

$\dfrac{d^2y}{dx^2} + P(x)\dfrac{du}{dx} + Q(x)u = x$, $x \in [-1,1]$

satisfying $u_1 \geq 0, u_2 \leq 0$ and $u_1(0) = u_2(0) = 0$. Let W denote the wronskian of u_1 and u_2 then

1. u_1 and u_2 are linearly independent.
2. u_1 and u_2 are linearly dependent.
3. $W(x) = 0$ for all $x \in [-1,1]$

4. $W(x) \neq 0$ for some $x \in [-1,1]$

11. Let $y:\mathbb{R} \to \mathbb{R}$ be a solution of ODE

$$\left.\begin{aligned} &\frac{d^2y}{dx^2} - y = e^{-x}, \ x \in \mathbb{R} \\ &y(0) = \frac{dy(0)}{dx} = 0 \end{aligned}\right\}$$

Then

1. y attains its minimum on $\mathbb{R}$
2. y is bounded on $\mathbb{R}$
3. $\lim_{x\to\infty} e^{-x} y(x) = \frac{1}{4}$
4. $\lim_{x\to-\infty} e^{-x} y(x) = \frac{1}{4}$

12. Let P be a continuous function on $\mathbb{R}$ and W be the wronskian of two linearly independent solutions y_1 and y_2 of the ODE :

$\frac{d^2y}{dx^2} + (1+x^2)\frac{dy}{dx} + P(x)y = 0, \ x \in \mathbb{R}$. Let $W(1) = a, W(2) = b$ and $W(3) = c$, then

1. $a < 0$ and $b > 0$
2. $a < b < c$ or $a > b > c$
3. $\frac{a}{|a|} = \frac{b}{|b|} = \frac{c}{|c|}$
4. $0 < a < b$ and $b > c > 0$

13. The problem $-y'' + (1+x)y = \lambda y, \ x \in (0,1)$ $y(0) = y(1) = 0$ has a non zero solution

1. for all $\lambda < 0$
2. for all $\lambda \in [0,1]$
3. for some $\lambda \in (2,\infty)$
4. for a countable number of λ's

14. Let $x:[0,3\pi] \to \mathbb{R}$ has a non zero solution of the ODE $x''(t) + e^{t^2} x(t) = 0,$ for $t \in [0,3\pi]$. Then the cardinality of the set $\{t \in [0,3\pi] : x(t) = 0\}$ is

1. equal to 1
2. greater than or equal to 2
3. equal to 2
4. greater than or equal to 3

15. Consider a boundary value problem $(BVP)\frac{d^2y}{dx^2} = f(x)$ with boundary conditions $y(0) = y(1) = y'(1)$, where f is a real-valued continuous function on $[0,1]$. Then which of the following are true ?

1. the given BVP has a unique solution for every f
2. the given BVP does not have a unique solution for some f
3. $y(x) = \int_0^x xt\, f(t)dt + \int_x^1 (t - x + xt) f(t) dt$ is a solution of the given BVP
4. $y(x) = \int_0^x (x - t + xt) f(t)dt + \int_x^1 xt\, f(t) dt$

(CSIR NET Dec 2017)

16. Consider the differential equation $\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} - y = 0$ defined on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Which among the following are true ?

1. there is exactly one solution $y = y(x)$ with $y(0) = y'(0) = 1$ and $y\left(\frac{\pi}{3}\right) = 2\left(1 + \frac{\pi}{3}\right)$
2. there is exactly one solution $y = y(x)$ with $y(0) = 1, y'(0) = -1$ and $y\left(-\frac{\pi}{3}\right) = 2\left(1 + \frac{\pi}{3}\right)$
3. any solution $y = y(x)$ satisfies

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$y''(0) = y(0)$

4. if y_1 and y_2 are any two solutions then $(ax+b)y_1 = (cx+d)y_2$ for some $a,b,c,d \in \mathbb{R}$

(CSIR NET Dec 2017)

17. Consider the Sturm-Liouville problem

$y''+\lambda y = 0,\ y(0) = 0$ and $y(\pi) = 0$.

Which of the following statements are true ?

1. There exist only countably many characteristic values
2. There exist uncountably many characteristic values
3. Each characteristic function corresponding to the characteristic value λ has exactly $\left[\sqrt{\lambda}\right]-1$ zeros in $(0,\pi)$
4. Each characteristic function corresponding to the characteristic value λ has exactly $\left[\sqrt{\lambda}\right]$ zeros in $(0,\pi)$

(CSIR NET June 2018)

Answers

------------------------ S C Q --------------------

1. 1	2. 3	3. 4	4. 1
5. 1	6. 4	7. 3	8. 1
9. 3	10. 4	11. 4	12. 1
13. 1	14. 1	15. 1	16. 1
17. 3	18. 4	19. 4	20. 3
21. 4			

---------------------- M C Q --------------------

1. 1,2,3,4	2. 2,3,4	3. 1,2,3
4. 1,2,3,4	5. 1,3	6. 1,2
7. 1,2,4	8. 1,2	9. 2
10. 2,3	11. 1,3	12. 2,3
13. 3,4	14. 2,4	15. 1,3
16. 1,2,3,4	17. 1,3	

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Chapter - 6

System of linear ordinary differential equations

Consider the homogeneous linear system

$$\frac{dx}{dt}=ax+by \text{ and } \frac{dy}{dt}=cx+dy \qquad ......(1)$$

where *a, b, c, d* are real constants.

Suppose $X=\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then above system can be expressed as $\frac{dx}{dt}=AX$ or $X'(t)=AX$

Working Rule to find the general solution of (1) :

Step (i) Find the eigen values of A by solving the characteristic equation $\lambda^2-(trA)\lambda+\det A=0$.

Suppose λ_1 and λ_2 are the eigen values of A.

Step (ii) We have the following different cases :

Case (a) : If λ_1 and λ_2 are real and distinct then find eigen vectors corresponding to λ_1 and λ_2, say these be v_1 and v_2. Then the general solution of (1) is given by $X(t)=c_1v_1e^{\lambda_1 t}+c_2v_2e^{\lambda_2 t}$. Here $v_1e^{\lambda_1 t}$ and $v_2e^{\lambda_2 t}$ are two L.I. solutions of (1).

Case (b): If λ_1 and λ_2 are complex and distinct, then case (a) is still valid. But a shorter method is given as : Suppose v_1 is an eigen vector corresponding to the eigen value λ_1, then one solution of system (1) is $v_1e^{\lambda_1 t}$. Let w_1 and w_2 be real and imaginary parts of $v_1e^{\lambda_1 t}$ then it can be proved that w_1 and w_2 are two L.I. solutions of (1) and so the general solution of (1) is given by $X(t)=c_1w_1+c_2w_2, c_1, c_2$ are arbitrary.

Case (c) : If $\lambda_1=\lambda_2=\lambda$ (real) and dimension of eigen space of λ is 2 then suppose v_1 and v_2 are two L.I. eigen vectors w.r.t. λ. Then $v_1e^{\lambda t}$ and $v_2e^{\lambda t}$ are two L.I. solutions of (1) and so the general solution of (1) is given by $X(t)=c_1v_1e^{\lambda t}+c_2v_2e^{\lambda t}$

Case (d) : If $\lambda_1=\lambda_2=\lambda$ (real) and dimension of eigen space of λ is one, then suppose one eigen vector is v_1. Now, construct a new vector v_2 which satisfy the equation $(A-\lambda I)v_2=v_1$. Then two L.I. solutions of (1) are given by $v_1e^{\lambda t}$ and $(tv_1+v_2)e^{\lambda t}$ and so the general solution is given by

$X(t)=c_1v_1e^{\lambda t}+c_2(tv_1+v_2)e^{\lambda t}$ where c_1, c_2 are arbitrary constants.

Autonomous Systems

Def. Autonomous System : Consider the system

$$\frac{dx}{dt} = F(x, y) \qquad \frac{dy}{dt} = G(x, y) \qquad ......(1)$$

where F and G are continuous and have continuous first partial derivative throughout the xy plane. A system of this kind in which the independent variable t does not appear in the function F and G is called an autonomous system.

Def. Solution and path of an autonomous system : Consider the autonomous system

$$\frac{dx}{dt} = F(x, y) \qquad \frac{dy}{dt} = G(x, y) \qquad ......(1)$$

By a theorem of system of differential equations , we have that for any given real number t_0 and any pair (x_0, y_0) of real numbers , there exists a unique solution $x = x(t)$, $y = y(t)$ ......(2) of the system (1) such that $x(t_0) = x_0$ and $y(t_0) = y_0$. If $x(t)$ and $y(t)$ are not both constant functions , then system (2) represents a curve in xy–plane which is called path (or orbit or trajectory) of the system (1).

Def. Critical point : Given the autonomous system

$$\frac{dx}{dt} = F(x, y) \quad \text{and} \quad \frac{dy}{dt} = G(x, y) \qquad ......(1)$$

a point (x_0, y_0) at which both $F(x_0, y_0) = 0$ and $G(x_0, y_0) = 0$ is called a critical point of (1). A critical point is also called equilibrium point or singular point.

Def. Isolated critical point : A critical point (x_0, y_0) of the autonomous system

$$\frac{dx}{dt} = F(x, y) \qquad \frac{dy}{dt} = G(x, y) \qquad ......(1)$$

is called an isolated critical point if there exist a circle $(x-x_0)^2 + (y-y_0)^2 = r^2$ about the point (x_0, y_0) such that (x_0, y_0) is the only critical point of (1) within this circle.

Note : For convenience , we shall take the critical point (x_0, y_0) to be the origin $(0, 0)$. There is no loss in generality in doing so , for if $(x_0, y_0) \neq (0, 0)$, then the translation of coordinates

$\xi = x - x_0$, $\eta = y - y_0$ transforms (x_0, y_0) into the origin in $\xi\,\eta$ plane.

Def. Let $x = x(t)$, $y = y(t)$ is a solution which prametrically represents the path C , and let $(0, 0)$ be a critical point of the autonomous system $\dfrac{dx}{dt} = F(x, y)$ $\dfrac{dy}{dt} = G(x, y)$.

Then we say that the path C approaches the critical point $(0, 0)$ as $t\to\infty$ if

$$\lim_{t\to\infty} x(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} y(t) = 0.$$

In like manner , a path C_1 approaches the critical point $(0, 0)$ as $t \to -\infty$ if

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$$\lim_{t\to-\infty} x_1(t) = 0 \qquad \text{and} \qquad \lim_{t\to-\infty} y_1(t) = 0$$

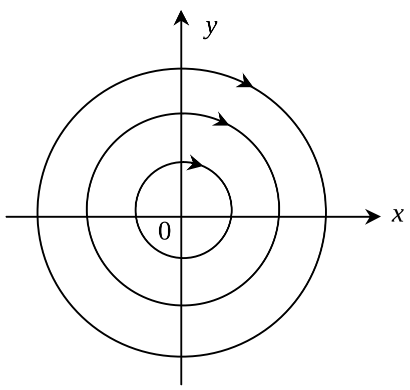
where $x = x_1(t)$ and $y = y_1(t)$ is a solution defining the path C_1.

Def. Let $x = x(t)$ and $y = y(t)$ be a solution which parametrically represents the path C and let $(0, 0)$ be the critical point of the autonomous system $\frac{dx}{dt} = F(x, y)$, $\frac{dy}{dt} = G(x, y)$ to which C approaches as $t \to \infty$. Then we say that C enters the critical point $(0, 0)$, as $t \to \infty$ if $\lim_{t\to\infty} \frac{y(t)}{x(t)}$ exists or if this quotient becomes either positively or negatively infinite as $t \to \infty$.

We observe that the quotient $\frac{y(t)}{x(t)}$ represents the slope of the line joining critical point $(0, 0)$ and a point R with coordinates $(x(t), y(t))$ on C. Thus when we say that a path C enters the critical point $(0, 0)$as $t \to \infty$ we mean that the line joining $(0, 0)$ and a point R tracing out C approaches a definite 'limiting' direction as $t \to \infty$.

Types of critical points

Def. Center : The isolated critical point (0, 0) of autonomous system is called a center if there exists a neighbourhood of (0, 0) which contains a countably infinite number of closed path C_n ($n = 1, 2, \ldots..$) , each of which contains $(0, 0)$ in its interior , and which are such that the diameters of the paths approach 0 as $n \to \infty$, but $(0, 0)$ is not approached by any path either as $t \to \infty$ or as $t \to -\infty$.



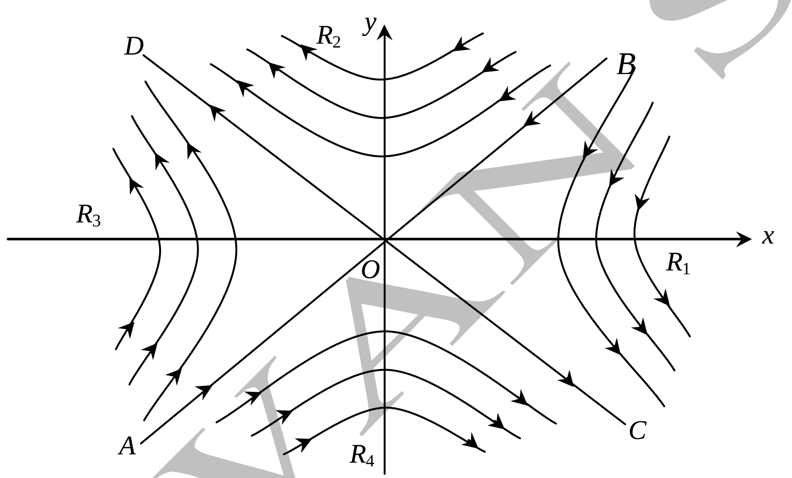
The critical point $(0, 0)$ of adjoining figure is called a center. Such a point is surrounded by an infinite family of closed paths , members of which are arbitrarily close to $(0, 0)$, but it is not approached by any path either as $t \to \infty$ or as $t \to -\infty$.

Def. Saddle Point : The isolated critical point (0 , 0) is called a saddle point if there exist a neighbourhood of (0 , 0) in which the following two conditions hold :

(i) There exist two paths which approach and enter (0, 0) from a pair of opposite directions as $t \to \infty$, and there exist two paths which approach and enter (0, 0) from a different pair of opposite directions as $t \to -\infty$.

(ii) In each of the four domains between any two of the four directions in (i) there are infinitely many paths which are arbitrarily close to (0 , 0) but which do not approach (0 , 0) either as $t \to \infty$ or as $t \to -\infty$.

The critical point (0 , 0) of adjoining figure is a saddle point which is such that



(i) It is approached and entered by two half-line paths (A_0 and B_0) as $t \to +\infty$, these two paths forming the geometric curve AB.

(ii) It is approached and entered by two half – line path (C_0 and D_0) as $t \to -\infty$, these two paths forming the geometric curve CD.

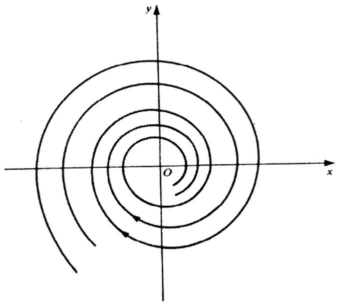
(iii) Between the four half – line paths described in (i) and (ii) there are four domains R_1 , R_3 , R_3 , R_4 , each containing an infinite family of semi – hyperbolic like paths which do not approach (0 , 0) as $t \to +\infty$ or as $t \to -\infty$, but which become asymptotic to one or another of the four half – line paths as $t \to \infty$ and as $t \to -\infty$.

Def. Focal point/Spiral point : The isolated critical point (0, 0) is called a spiral point (or focal point) if there exists a neighbourhood of (0 , 0) such that every path C in this neighbourhood has the following properties :

(i) C is defined for all $t > t_0$ (or for all $t < t_0$) for some number t_0.

(ii) C approaches (0 , 0) as $t \to +\infty$ (or as $t \to -\infty$).

(iii) C approaches (0 , 0) in a spiral – like manner , winding around (0 , 0) an infinite number of times as $t \to +\infty$ (or as $t \to -\infty$).



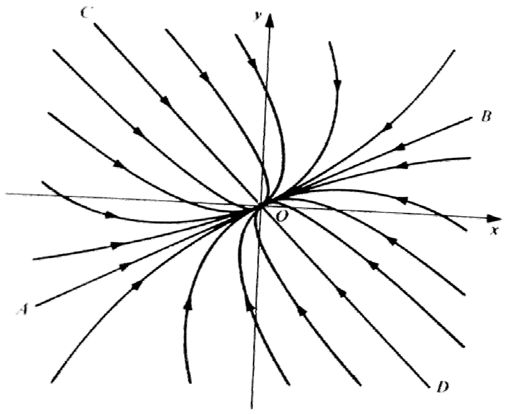
The critical point (0 , 0) in the above figure is a spiral point (or focal point). This point is approached in a spiral – like manner by an infinite family of paths as $t \to \infty$ (or as $t \to -\infty$). Here , while the paths approach (0, 0) , they do not enter it. That is a point R tracing such a path C approaches O (0 , 0) as $t \to \infty$ (or as $t \to -\infty$), but the line OR does not tend to a definite direction , since the path constantly winds about O.

Def. Node : The isolated critical point (0 , 0) is called a node if there exist a neighbourhood of (0 , 0) such that every path C in this neighbourhood has the following properties :

(i) C is defined for all $t > t_0$ (or for all $t < t_0$) for some number t_0.

(ii) C approaches (0 , 0) as $t \to +\infty$ (or as $t \to -\infty$).

(iii) C enters (0 , 0) as $t \to +\infty$ [or as $t \to -\infty$].



The critical point (0 , 0) in the above figure is a node. This point is not only approached but also entered by an infinite family of paths as $t \to \infty$ (or as $t \to -\infty$). That is , a point R tracing such a path not only approaches O but does so in such a way that the line OR tends to a definite direction as $t \to +\infty$ (or as $t \to -\infty$). In above figure , there are four rectilinear paths AO , BO , CO and DO. All other paths are like "semiparabolas" As each of these semiparabolic – like paths approaches O , its slope approaches that of the line AB.

Def. Stability: Let (0 , 0) is an isolated critical point of the autonomous system

$$\frac{dx}{dt} = F(x, y), \quad \frac{dy}{dt} = G(x, y) \qquad \text{.......(1)}$$

Let C be a path of (1) and let $x = x(t)$, $y = y(t)$ be a solution of (1) defining C parametrically.

Let

$$D(t) = \sqrt{[x(t)]^2 + [y(t)]^2} \qquad ......(2)$$

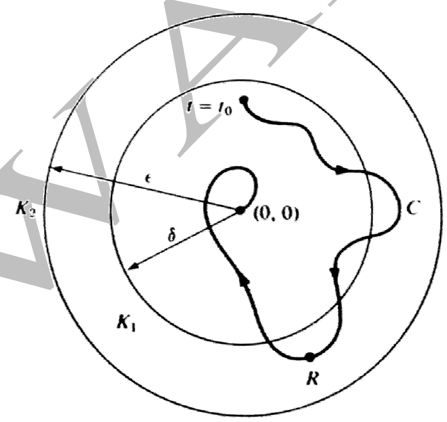
denote the distance between the critical point (0 , 0) and the point $R : (x(t) , y(t))$ on C. The critical point (0 , 0) is called stable if for every number $\varepsilon > 0$, there exists a number $\delta > 0$ such that the following is true : Every path C for which

$$D(t_0) < \delta \quad \text{for some value } t_0 \qquad ......(3)$$

is defined for all $t \geq t_0$ and is such that

$$D(t) < \varepsilon \quad \text{for } t_0 \leq t < \infty \qquad ......(4)$$

Analysis of the definition : According to (2) , the inequality $D(t_0) < \delta$ for some value t_0 in (3) , means that the distance between the critical point (0 , 0) and the point R on the path C must be less than δ at $t = t_0$. This means that at $t = t_0$, R lies within the circle K_1 of radius δ about (0 , 0). Similarly the inequality $D(t) < \varepsilon$ for $t_0 \leq t < \infty$ in (4) means that the distance between (0 , 0) and R is less than ε for all $t \geq t_0$, and hence that for $t \geq t_0$, R lies within the circle K_2 of radius ε about (0 , 0). Now if (0 , 0) is stable , then every path C which is inside the circle K_1 of radius δ at $t = t_0$ will remain inside the circle K_2 of radius $t \geq t_0$.



Asymptotic Stability : Let (0 , 0) is an isolated critical point of the system

$$\frac{dx}{dt} = F(x , y) , \quad \frac{dy}{dt} = G(x , y) \qquad ......(1)$$

Let C be a path of (1) and let $x = x(t)$, $y = y(t)$ be a solution of (1) representing C parametrically. Let

$$D(t) = \sqrt{[x(t)]^2 + [y(t)]^2} \qquad ....(2)$$

denote the distance between the critical point (0 , 0) and the point $R : (x(t) , y(t))$ on C. The critical point (0 , 0) is called asymptotically stable if

(i) It is stable and

(ii) There exist a number $\delta_0 > 0$ such that if

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$$D(t_0) < \delta_0 \qquad \text{......(3)}$$

for some value t_0 , then

$$\lim_{t\to\infty} x(t) = 0 \quad , \quad \lim_{t\to\infty} y(t) = 0 \qquad \text{......(4)}$$

Def. A critical point is called unstable if it not stable.

Illustration : (i) Centre , spiral point and node are stable.

(ii) Out of these three , the spiral point and the node are asymptotically stable.

(iii) If the directions of the paths in the figures of spiral point and node are reversed , then they become unstable.

(iv) Saddle point is unstable.

Critical points and paths of linear systems : We consider the linear system

$$\left.\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned}\right\} \qquad \text{.......(1)}$$ where a , b , c , d are real constants.

We attempt to determine a solution of the form

$$\left.\begin{aligned} x &= A\, e^{\lambda t} \\ y &= B\, e^{\lambda t} \end{aligned}\right\} \qquad \text{.......(2)}$$ where A , B and λ are constants.

If we put (2) in (1) , we obtain

$$A\lambda\, e^{\lambda t} = a A\, e^{\lambda t} + b B\, e^{\lambda t}$$

$$B\lambda\, e^{\lambda t} = c A\, e^{\lambda t} + d B e^{\lambda t}$$ which gives

$$\left.\begin{aligned} (a-\lambda)A + bB &= 0 \\ a A + (b-\lambda) B &= 0 \end{aligned}\right\} \qquad \text{.......(3)}$$

This system obviously has the trivial solution $A = B = 0$. But this would give only the trivial solution $x = 0$, $y = 0$ of the system (1). Thus we seek non – trivial solution of the system (3). A necessary and sufficient condition that this system have a non-trivial solution is that the determinant

$$\begin{vmatrix} a-\lambda & b \\ a & d-\lambda \end{vmatrix} = 0 \qquad \text{......(4)}$$

This gives the quadratic equation

$$\lambda^2 - (a+d)\,\lambda + (ad - bc) = 0 \qquad \text{......(5)}$$

This equation is called the characteristic equation associated with the system (1). Its roots λ_1 and λ_2 are called the characteristic roots.

If the pair (2) is to be a solution of the system (1) , then λ in (2) must be one of these roots.

Note : We summarize our results in the following table :

Sr. No.	Nature of Roots λ_1 , λ_2	Nature of Critical point	Stability of critical point
1.	Real, unequal, same sign	Node	Asymptotically stable if roots are negative and unstable if roots are positive.
2.	Real,unequal, opposite sign	Saddle point	Unstable.
3.	Real and equal.	Node	Asymptotically stable if roots are negative and unstable if roots are positive.
4.	Conjugate complex but not pure imaginary.	Spiral point	Asymptotically stable if real part of roots is negative and unstable if real part of roots is positive.
5.	Pure imaginary	Center	Stable but not asymptotically stable.

Example 1 : Determine the nature of the critical point (0 , 0) of the system

$$\left.\begin{aligned} \frac{dx}{dt} &= 2x - 7y \\ \frac{dy}{dt} &= 3x - 8y \end{aligned}\right] \qquad \text{.......(1)}$$

and determine whether or not the point is stable.

Solution : Comparing system (1) with standard system

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy$$

we get , $a = 2$, $b = -7$, $c = 3$ and $d = -8$.

We know that characteristic equation is $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$

i.e $\quad \lambda^2 + 6\lambda + 5 = 0 \qquad$ ......(2)

Hence the roots of characteristic equation are $\lambda_1 = -5$ and $\lambda_2 = -1$. Since the roots are real ,

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unequal and of the same sign (both negative) , we conclude that the critical point (0 , 0) of (1) is a node. Since the roots are real and negative , the point is asymptotically stable.

Example 2 : Determine the nature of the critical point (0, 0) of the system

$$\left.\begin{aligned}\frac{dx}{dt} &= 2x + 4y\\ \frac{dy}{dt} &= -2x + 6y\end{aligned}\right] \quad \text{......(1)}$$

and determine whether or not the point is stable.

Solution : Here $a = 2$, $b = 4$, $c = -2$, $d = 6$. The characteristic equation is

$$\lambda^2 - 8\lambda + 20 = 0$$

and its roots are $4 \pm 2i$. Since these roots are conjugate complex but not pure imaginary, so critical point (0 , 0) is a spiral point. Since the real part of the conjugate complex roots is positive , the point is unstable.

Exercise 6.1

Find the general solution of each of the following linear systems:

(1) $\frac{dx}{dt} = 5x - 2y, \frac{dy}{dt} = 4x - y$ (2) $\frac{dx}{dt} = x + 2y, \frac{dy}{dt} = 3x + 2y$ (3) $\frac{dx}{dt} = 3x + y, \frac{dy}{dt} = 4x + 3y$

(4) $\frac{dx}{dt} = 3x - 4y, \frac{dy}{dt} = 2x - 3y$ (5) $\frac{dx}{dt} = x + 3y, \frac{dy}{dt} = 3x + y$ (6) $\frac{dx}{dt} = x - 4y, \frac{dy}{dt} = x + y$

(7) $\frac{dx}{dt} = x - 3y, \frac{dy}{dt} = 3x + y$ (8) $\frac{dx}{dt} = 4x - 2y, \frac{dy}{dt} = 5x + 2y$ (9) $\frac{dx}{dt} = 3x - 2y, \frac{dy}{dt} = 2x + 3y$

(10) $\frac{dx}{dt} = 3x - y, \frac{dy}{dt} = 4x - y$ (11) $\frac{dx}{dt} = 5x + 4y, \frac{dy}{dt} = -x + y$

Find the particular solution of each of the following linear systems:

(12) $\frac{dx}{dt} = -2x + 7y, \frac{dy}{dt} = 3x + 2y; x(0) = 9, y(0) = -1$

(13) $\frac{dx}{dt} = 2x - 8y, \frac{dy}{dt} = x + 6y; x(0) = 4, y(0) = 1$

(14) $\frac{dx}{dt} = 6x - 4y, \frac{dy}{dt} = x + 2y; x(0) = 2, y(0) = 3$

Find the general solution of each of the following homogeneous linear systems, where $X=\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ has been used.

(15) $\dfrac{dX}{dt}=\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}X$ (16) $\dfrac{dX}{dt}=\begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}X$ (17) $\dfrac{dX}{dt}=\begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}X$ (18) $\dfrac{dX}{dt}=\begin{bmatrix} 6 & -4 \\ 1 & 2 \end{bmatrix}X$

Answers

1. $x_1=c_1e^t+c_2e^{3t},\ x_2=2c_1e^t+c_2e^{3t}$
2. $x_1=c_1e^{-t}+2c_2e^{4t},\ x_2=-c_1e^{-t}+3c_2e^{4t}$
3. $x_1=c_1e^t+c_2e^{5t},\ x_2=-2c_1e^t+2c_2e^{5t}$
4. $x_1=2c_1e^t+c_2e^{-t},\ x_2=c_1e^t+c_2e^{-t}$
5. $x_1=c_1e^{-2t}+c_2e^{4t},\ x_2=-c_1e^{-2t}+c_2e^{4t}$
6. $x_1=-2c_1\cos 2t\,e^t-2c_2\sin 2t\,e^t,\ x_2=-c_1\sin 2t\,e^t+c_2\cos 2t\,e^t$
7. $x_1=c_1\cos 3t\,e^t+c_2\sin 3t\,e^t,\ \ x_2=c_1\sin 3t\,e^t-c_2\cos 3t\,e^t$
8. $x_1=2c_1\cos 3t\,e^{3t}+c_2e^{3t}(\cos 3t+3\sin 3t),\ x_2=2c_1e^{3t}\sin 3t+c_2e^{3t}(\sin 3t-3\cos 3t)$
9. $x_1=-c_1e^{3t}\sin 2t+c_2e^{3t}\cos 2t,\ x_2=c_1e^{3t}\cos 2t+c_2e^{3t}\sin 2t$
10. $x_1=c_1e^t+c_2(t+1)e^t,\ x_2=2c_1e^t+c_2(2t+1)e^t$
11. $x_1=2c_1e^{3t}+c_2(2t+1)e^{3t},\ x_2=-c_1e^{3t}-c_2te^{3t}$
12. $x_1=2e^{5t}+7e^{-5t},\ x_2=2e^{5t}-3e^{-5t}$
13. $x_1=(4\cos 2t-8\sin 2t)e^{4t},\ x_2=(\cos 2t+3\sin 2t)e^{4t}$
14. $x_1=(2-8t)e^{4t},\ x_2=(3-4t)e^{4t}$
15. $x_1=c_1e^{-t}+2c_2e^{4t},\ x_2=-c_1e^{-t}+3c_2e^{4t}$
16. $x_1=-2c_1e^t\sin 2t+2c_2e^t\cos 2t,\ x_2=c_1e^t\cos 2t+c_2e^t\sin 2t$
17. $x_1=(c_1+tc_2)e^t,\ \ x_2=[2c_1+c_2(2t-1)]e^t$
18. $x_1=2c_1e^{4t}+c_2(2t+1)e^{4t},\ x_2=c_1e^{4t}+c_2te^{4t}$

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Critical points and paths of non-linear system

Theorem (A) : Basic theorems on non-linear system : We now consider the non-linear real autonomous system

$$\begin{aligned}\frac{dx}{dt} &= P(x,y)\\ \frac{dy}{dt} &= Q(x,y)\end{aligned} \qquad \ldots\ldots(*)$$

We assume that the system (*) has an isolated critical point which we shall choose to be origin (0, 0). We now further assume that the functions P and Q in the right members of (*) are such that $P(x,y)$ and $Q(x,y)$ can be written in the form

$$\begin{aligned}P(x,y) &= ax+by+P_1(x,y)\\ Q(x,y) &= cx+dy+Q_1(x,y)\end{aligned} \qquad \ldots..(**)$$

where

I. a,b,c,d are real constants and $\begin{vmatrix} a & b\\ c & d\end{vmatrix} \neq 0$ and

II. P_1 and Q_1 have continuous first partial derivatives for all (x,y) and are such that

$$\lim_{(x,y)\to(0,0)} \frac{P_1(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{Q_1(x,y)}{\sqrt{x^2+y^2}} = 0 \qquad \ldots..(1)$$

Thus the system under consideration may be written in the form

$$\begin{aligned}\frac{dx}{dt} &= ax+by+P_1(x,y)\\ \frac{dy}{dt} &= cx+dy+Q_1(x,y)\end{aligned}$$

where a,b,c,d,P_1 and Q_1 satisfy the requirements (1) and (II)

Theorem (B) : Consider the non linear system

$$\begin{aligned}\frac{dx}{dt} &= ax+by+P_1(x,y)\\ \frac{dy}{dt} &= cx+dy+Q_1(x,y)\end{aligned} \qquad \ldots..(2)$$

where a,b,c,d,P_1 and Q_1 satisfy the requirements (1) and (II) above consider also the corresponding linear system

$$\begin{aligned}\frac{dx}{dt} &= ax+by\\ \frac{dy}{dt} &= cx+dy\end{aligned} \qquad \ldots.(3)$$

obtained from (2) by neglecting the nonlinear terms $P_1(x, y)$ and $Q_1(x, y)$. Both systems have an isolated critical point at (0, 0). Let λ_1 and λ_2 be the roots of the characteristic equation

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0 \qquad \text{……(4)}$$

of the linear system (3).

Conclusions : 1. The critical point (0, 0) of the nonlinear system (2) is of the same type as that of the linear system (3) in the following cases :

Sr. No.	Nature of root λ_1, λ_2	Nature of critical point	Stability of critical point
1.	Real, unequal, same sign	(0, 0) is a node for both system (2) and (3)	Asymptotically stable if roots are negative and unstable if roots are positive
2.	Real, unequal, opposite sign	(0, 0) is a saddle point for both system (2) and (3)	Unstable
3.	Real, equal and system is not such that $a = d \neq 0,\ b = c = 0$	(0, 0) is a node for both systems (2) and (3)	Asymptotically stable if roots are negative and unstable if roots are positive.
4.	Real, equal and system (3) is such that $a = d \neq 0,\ b = c = 0$	(0, 0) is a node of (3) but (0, 0) may be either a node or a spiral point of (2)	Asymptotically stable if roots are negative and unstable if roots are positive.
5.	Pure imaginary	(0, 0) is a centre of (3) but (0, 0) may be either a centre or a spiral point of (2).	(0, 0) is stable critical point of (3), it is not necessarily a stable critical point of (2).
6.	Conjugate complex but not pure imaginary	(0, 0) is a spiral point for both systems (2) and (3)	Asymptotically stable if real part of roots is negative and unstable if real part of roots is positive.

Example 1 : Determine the nature of the critical point (0, 0) of the system $\left.\begin{aligned}\frac{dx}{dt} &= 8x - y^2 \\ \frac{dy}{dt} &= -6y + 6x^2\end{aligned}\right]$ ……(1)

and determine whether or not the point is stable.

Solution : System (1) is of the form (2) and that the hypotheses of theorem (A) and (B) are satisfied. To determine the type of critical point (0, 0), we consider the linear system

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$$\frac{dx}{dt} = 8x$$

$$\frac{dy}{dt} = -6y$$

of the form (3). The characteristic equation of this system is $\lambda^2 - 2\lambda - 48 = 0$ and thus the roots are $\lambda_1 = 8$ and $\lambda_2 = -6$. Since roots are real, unequal and of opposite sign, so critical point (0, 0) of the given nonlinear system is a saddle point. So it is unstable. By definition, the critical points of this system must simultaneously satisfy the system of algebraic equations $8x - y^2 = 0$, $-6y + 6x^2 = 0$
From the second equation of this pair, $y - x^2$. Then, substituting this into the first equation of the pair, we obtain $8x - x^4 = 0$, which factors into $x(2-x)(4+2x+x^2) = 0$
This equation has only two real roots $x = 0$ and $x = 2$. Thus we obtain two real critical points (0, 0), and (2, 4).

We now investigate the type and stability of the other critical point (2, 4). To do this we make the translation of coordinates $\xi = x - 2, \quad \eta = y - 4$ which transforms the critical point $x = 2, \quad y = 4$ into the origin $\xi = \eta = 0$ in the $\xi\eta$ plane. We now transforms the given system into (ξ, η) coordinates.

$$\frac{d\xi}{dt} = 8\xi - 8\eta - \eta^2 \qquad \text{.......(2)}$$

$$\frac{d\eta}{dt} = -24\xi - 6\eta + 6\xi^2$$

The system (2) is of the form (2) and hypotheses of theorem (A) and (B) are satisfied in these coordinates. To determine the type of the critical point $\xi = \eta = 0$ of (2), we consider the system

$$\frac{d\xi}{dt} = 8\xi - 8\eta$$

$$\frac{d\eta}{dt} = 24\xi - 6\eta$$

The characteristic equation of this linear system is $\lambda^2 - 2\lambda + 144 = 0$
The roots of this system are $1 \pm \sqrt{143}\,i$ which are conjugate complex with real part not zero, so $\xi = \eta = 0$ of non linear system (2) is a spiral point and since real part is positive so it is a unstable spiral point. So the given system (1) has two real critical point namely

1. critical point (0, 0) ; a saddle point ; unstable.
2. critical point (2, 4) ; a spiral point ; unstable.

Exercise 6.2

1. Determine the nature of the critical point (0, 0) of the system

$$\frac{dx}{dt} = x + 4y - x^2$$

$$\frac{dy}{dt} = 6x - y + 2xy$$

   and determine whether or not the point is stable.

2. Determine the nature of critical point (0, 0) of the system

$$\frac{dx}{dt} = \sin x - 4y$$

$$\frac{dy}{dt} = \sin 2x - 5y$$

   and determine whether or not the point is stable.

Answers

1. saddle point ; unstable
2. node ; asymptotically stable

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Assignment-6

--------------------- S C Q ---------------------

1. The general solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ of the system
$$\begin{aligned} x' &= -x+2y \\ y' &= 4x+y \end{aligned}$$ is given by
   1. $\begin{pmatrix} c_1e^{3t}-c_2e^{-3t} \\ 2c_1e^{3t}+c_2e^{-3t} \end{pmatrix}$
   2. $\begin{pmatrix} c_1e^{3t} \\ c_2e^{-3t} \end{pmatrix}$
   3. $\begin{pmatrix} c_1e^{3t}+c_2e^{-3t} \\ 2c_1e^{3t}+c_2e^{-3t} \end{pmatrix}$
   4. $\begin{pmatrix} c_1e^{3t}-c_2e^{-3t} \\ -2c_1e^{3t}+c_2e^{-3t} \end{pmatrix}$

(GATE 2001)

2. The general solution of the system of differential equations
$$y+\frac{dz}{dx}=0$$
$$\frac{dy}{dx}-z=0$$
is given by
   1. $y=\alpha e^{x}+\beta e^{-x},\ z=\alpha e^{x}-\beta e^{-x}$
   2. $y=\alpha\cos x+\beta\sin x,\ z=\alpha\sin x-\beta\cos x$
   3. $y=\alpha\sin x-\beta\cos x,\ z=\alpha\cos x+\beta\sin x$
   4. $y=\alpha e^{x}-\beta e^{-x},\ z=\alpha e^{x}+\beta e^{-x}$

(GATE 2005)

3. Let $a,b\in R$. Let $y=(y_1,y_2)^t$ be a solution of the system of equation
$y_1'=y_2,\ y_2'=ay_1+by_2$.
Every solution $y(x)\to 0$ as $x\to\infty$, if
   1. $a<0, b<0$ 2. $a<0, b>0$
   3. $a>0, b>0$ 4. $a>0, b<0$

(GATE 2008)

4. Let $y(t)=\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ satisfy $\frac{dy}{dt}=Ay;\ t>0$ and $y(0)=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ where A is a 2×2 constant matrix with real entries satisfying trace A=0 and det$A$>0. Then $y_1(t)$ and $y_2(t)$ both are
   1. monotonically decreasing functions of t.
   2. monotonically increasing functions of t.
   3. oscillating functions of t.
   4. constant functions of t.

(CSIR NET Dec 2012)

5. Consider the system of ODE in $\mathbb{R}^2, \frac{dy}{dt}=AY, Y(0)=\begin{pmatrix} 0 \\ 1 \end{pmatrix}, t>0$ where $A=\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ and $Y(t)=\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$.
Then
   1. $y_1(t)$ and $y_2(t)$ are monotonically increasing for $t>0$
   2. $y_1(t)$ and $y_2(t)$ are monotonically increasing for $t>1$
   3. $y_1(t)$ and $y_2(t)$ are monotonically decreasing for $t>0$
   4. $y_1(t)$ and $y_2(t)$ are monotonically decreasing for $t>1$

(CSIR NET Dec 2015)

6. Let $A=\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}, x(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$ and
$|x(t)|=(x_1^2(t)+x_2^2(t)+x_3^2(t))^{1/2}$
Then any solution of the first order system of the ordinary differential equation
$\left.\begin{aligned} x'(t)&=Ax(t) \\ x(0)&=x_0 \end{aligned}\right\}$ satisfies
   1. $\lim_{t\to\infty}|x(t)|=0$

2. $\lim_{t\to\infty}|x(t)|=\infty$
3. $\lim_{t\to\infty}|x(t)|=2$
4. $\lim_{t\to\infty}|x(t)|=12$

(CSIR NET June 2016)

---------------------- **M C Q** --------------------

1. Consider the first order system of linear equations $\frac{dX}{dt}=AX$; $A=\begin{bmatrix}3 & 2\\ -2 & -1\end{bmatrix}$; $X=\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$. Then
   1. the coefficient matrix A has a repeated eigenvalue $\lambda=1$
   2. there is only one linearly independent eigenvector $X_1=\begin{bmatrix}1\\ -1\end{bmatrix}$
   3. the general solution of the ODE is $(aX_1+bX_2)e'$, where a, b are arbitrary constants and $X_1=\begin{bmatrix}1\\ -1\end{bmatrix}, X_2=\begin{bmatrix}t\\ \frac{1}{2}-t\end{bmatrix}$
   4. the vectors X_1 and X_2 in the option 3 given above are linearly independent

(CSIR NET Dec 2011)

2. Consider the system of ODE $\frac{d}{dx}Y=AY, Y(0)=\begin{bmatrix}2\\ -1\end{bmatrix}$; where $A=\begin{pmatrix}1 & 2\\ 0 & -1\end{pmatrix}$ and $Y=\begin{bmatrix}y_1(x)\\ y_2(x)\end{bmatrix}$. Then
   1. $y_1(x)\to\infty$ and $y_2(x)\to 0$ as $x\to\infty$
   2. $y_1(x)\to 0$ and $y_2(x)\to 0$ as $x\to\infty$
   3. $y_1(x)\to\infty$ and $y_2(x)\to\infty$ as $x\to-\infty$
   4. $y_1(x), y_2(x)\to-\infty$ as $x\to-\infty$

(CSIR NET June 2012)

3. Let $A=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$ and $y=\begin{pmatrix}y_1(t)\\ y_2(t)\\ y_3(t)\end{pmatrix}$ satisfy $\frac{dy}{dt}=Ay; t>0; y(0)=\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$. Then
   1. $y_1(t)=1+t+\frac{t^2}{2}, y_2(t)=1+t, y_3(t)=1.$
   2. $y_1(t)=1+t, y_2(t)=1+t+\frac{t^2}{2}, y_3(t)=1.$
   3. $y_1(t)=1, y_2(t)=1+t, y_3(t)=1+t+\frac{t^2}{2}.$
   4. $y_1(t)=e^{tA}y(0).$

(CSIR NET Dec 2013)

4. The critical point of the system $\frac{dy}{dx}=-4x-y,\ \frac{dy}{dt}=x-2y$ is an
   1. asymptotically stable node
   2. unstable node
   3. asymptotically stable spiral
   4. unstable spiral

(CSIR NET June 2015)

5. Let $(x(t), y(t))$ satisfy for $t>0$ $\frac{dx}{dt}=-x+y,\ \frac{dy}{dt}=-y,\ x(0)=y(0)=1,$ Then $x(t)$ is equal to
   1. $e^{-t}+t\,y(t)$
   2. $y(t)$
   3. $e^{-t}(1+t)$
   4. $-y(t)$

(CSIR NET Dec 2016)

6. Consider a system of first order differential equations $\frac{d}{dt}\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}=\begin{bmatrix}x(t)+y(t)\\ -y(t)\end{bmatrix}$ The solution space is spanned by
   1. $\begin{bmatrix}0\\ e^{-t}\end{bmatrix}$ and $\begin{bmatrix}e^{t}\\ 0\end{bmatrix}$
   2. $\begin{bmatrix}e^{t}\\ 0\end{bmatrix}$ and $\begin{bmatrix}\cosh t\\ e^{-t}\end{bmatrix}$
   3. $\begin{bmatrix}e^{-t}\\ -2e^{-t}\end{bmatrix}$ and $\begin{bmatrix}\sinh t\\ e^{-t}\end{bmatrix}$

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4. $\begin{bmatrix} e^t \\ 0 \end{bmatrix}$ and $\begin{bmatrix} e^t - \frac{1}{2}e^{-t} \\ e^{-t} \end{bmatrix}$

7. Consider the system of differential equations

$$\frac{dx}{dt} = 2x - 7y$$
$$\frac{dy}{dt} = 3x - 8y$$

Then the critical point $(0,0)$ of the system is an

1. asymptotically stable node
2. unstable node
3. asymptotically stable spiral
4. unstable spiral

(CSIR NET June 2018)

Answers

----------------------- S C Q --------------------

1. 1	2. 3	3. 1
4. 3	5. 4	6. 1

---------------------- M C Q --------------------

1. 1,2,3,4	2. 1	3. 1
4. 1	5. 1,3	6. 3,4
7. 1		

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Chapter - 7

Existence and uniqueness problems

Def. Lipschitz condition : A function $f(x,y)$ defined in domain D is said to satisfy the Lipschitz condition w.r.t. y in D if, these exists constant K such that $\left|f(x,y_1)-f(x,y_2)\right|\le K\left|y_1-y_2\right|$, for all $(x,y_1),(x,y_2)\in D$. Here the constant K is known as Lipschitz constant.

Theorem (1) : Suppose $f(x,y)$ is a function defined and continuous in a domain D such that $\frac{\partial f}{\partial y}$ exists and bounded in D then $f(x,y)$ satisfy the Lipschitz condition with the Lipschitz constant

$\max\left|\frac{\partial f}{\partial y}\right|,(x,y)\in D$.

Remark: Boundness of $\frac{\partial f}{\partial y}$ is only a sufficient condition for the *Lip*. Condition but not necessary i.e. If $\frac{\partial f}{\partial y}$ is not bounded then nothing can be said about Lipschitz condition.

Theorem (2) : Cauchy–Peano Existence theorem :

Hypothesis : Consider the initial value problem

$$\frac{dy}{dx}=f(x,y) \qquad \text{......(1)}$$

and $$y(x_0)=y_0 \qquad \text{......(2)}$$

Suppose $f(x,y)$ is continuous in the closed rectangle R defined by

$$R=\{(x,y):|x-x_0|\le a,\ |y-y_0|\le b\}$$

Conclusion : (i) These exist a solution $\phi(x)$ of (1) and (2).

(ii) This solution $\phi(x)$ is defined at least on the interval $[x_0-h,x_0+h]$, where $h=\min\left\{a,\frac{b}{M}\right\}$ and

$|f(x,y)\le M$ for all $x\in R$

Theorem (3) : Picard's Uniqueness theorem :

Hypothesis : Consider the I.V.P.

$$\frac{dy}{dx} = f(x,y) \qquad \text{......(1)}$$

and $$y(x_0) = y_0 \qquad \text{......(2)}$$

Suppose the function $f(x,y)$ is (C, Lip) in the rectangle $R = \{(x,y) : |x - x_0| \le a, |y - y_0| \le b\}$

Conclusion : (i) These exist a unique solution $\phi(x)$ of (1) and (2).

(ii) This solution $\phi(x)$ is defined atleast on the interval $[x_0 - h, x_0 + h]$, where $h = \min\left\{a, \frac{b}{M}\right\}$ and

$|f(x,y)| \le M$ for all $x \in R$

Remark : If the function $f(x,y)$ is not (C, Lip) then nothing can be said about existence and uniqueness of the solution.

Exercise 7.1

1. Which of the following functions satisfies a Lipschitz condition in the rectangle R defined by $R = \{(x,y) : |x| \le a, |y| \le b, a > 0, b > 0\}$

(i) $f(x,y) = x^2 + y^2$ (ii) $f(x,y) = x \sin y + y \cos x$ (iii) $f(x,y) = x^2 e^{x+y}$

(iv) $f(x,y) = xy$ (v) $f(x,y) = y^2 + y^3$ (vi) $f(x,y) = y^{\frac{1}{3}}$

(vii) $f(x,y) = y^{\frac{2}{3}}$ (viii) $f(x,y) = y^{\frac{2}{5}}$ (ix) $f(x,y) = y^{\frac{3}{2}}$

(x) $f(x,y) = y^{\frac{5}{2}}$ (xi) $f(x,y) = xy^{\frac{2}{3}}$ (xii) $f(x,y) = xy^{\frac{3}{2}}$

(xiii) $f(x,y) = \sqrt{y}$ (xiv) $f(x,y) = x\sqrt{y}$ (xv) $f(x,y) = \sqrt{|y|}$

(xvi) $f(x,y) = \sqrt{|xy|}$ (xvii) $f(x,y) = |y|^{\frac{3}{5}}$ (xviii) $f(x,y) = |y|^{\frac{5}{3}}$

(xix) $f(x,y) = |\sin y|$ (xx) $f(x,y) = (1 + x^2)y$

(xxi) $f(x,y) = g(x)y$ where $g(x)$ is continuous on R.

(xxii) $f(x,y) = (1 + g^2(x))y$, where $g(x)$ is continuous on R.

(xxiii) $f(x,y) = a_0(x)y^2 + a_1(x)y + a_2(x)$, where a_0, a_1 and a_2 are continuous functions on R.

(xxiv) $f(x,y) = x|y|$

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2. Which of the following functions satisfies a Lipschitz condition in the rectangle R defined by $R=\{(x,y):|x-3|\le 1, |y|\le 2\}$.

(i) $f(x,y)=x^2+y^2$ (ii) $f(x,y)=x\sin y+y\cos x$ (iii) $f(x,y)=x^2e^{x+y}$

(iv) $f(x,y)=xy$ (v) $f(x,y)=y^2+y^3$ (vi) $f(x,y)=y^{\frac{1}{3}}$

(vii) $f(x,y)=y^{\frac{2}{3}}$ (viii) $f(x,y)=y^{\frac{2}{5}}$ (ix) $f(x,y)=y^{\frac{3}{2}}$

(x) $f(x,y)=y^{\frac{5}{2}}$ (xi) $f(x,y)=xy^{\frac{2}{3}}$ (xii) $f(x,y)=xy^{\frac{3}{2}}$

(xiii) $f(x,y)=\sqrt{y}$ (xiv) $f(x,y)=x\sqrt{y}$ (xv) $f(x,y)=\sqrt{|y|}$

(xvi) $f(x,y)=\sqrt{|xy|}$ (xvii) $f(x,y)=|y|^{\frac{3}{5}}$ (xviii) $f(x,y)=|y|^{\frac{5}{3}}$

(xix) $f(x,y)=|\sin y|$ (xx) $f(x,y)=(1+x^2)y$

(xxi) $f(x,y)=g(x)y$ where $g(x)$ is continuous on R.

(xxii) $f(x,y)=(1+g^2(x))y$, where $g(x)$ is continuous on R.

(xxiii) $f(x,y)=a_0(x)y^2+a_1(x)y+a_2(x)$, where a_0, a_1 and a_2 are continuous functions on R.

(xxiv) $f(x,y)=x|y|$

3. Which of the following functions satisfies a Lipschitz condition in the rectangle R defined by $R=\{(x,y):|x-3|\le 1, |y-3|\le 2\}$.

(i) $f(x,y)=x^2+y^2$ (ii) $f(x,y)=x\sin y+y\cos x$ (iii) $f(x,y)=x^2e^{x+y}$

(iv) $f(x,y)=xy$ (v) $f(x,y)=y^2+y^3$ (vi) $f(x,y)=y^{\frac{1}{3}}$

(vii) $f(x,y)=y^{\frac{2}{3}}$ (viii) $f(x,y)=y^{\frac{2}{5}}$ (ix) $f(x,y)=y^{\frac{3}{2}}$

(x) $f(x,y)=y^{\frac{5}{2}}$ (xi) $f(x,y)=xy^{\frac{2}{3}}$ (xii) $f(x,y)=xy^{\frac{3}{2}}$

(xiii) $f(x,y)=\sqrt{y}$ (xiv) $f(x,y)=x\sqrt{y}$ (xv) $f(x,y)=\sqrt{|y|}$

(xvi) $f(x,y)=\sqrt{|xy|}$ (xvii) $f(x,y)=|y|^{\frac{3}{5}}$ (xviii) $f(x,y)=|y|^{\frac{5}{3}}$

(xix) $f(x,y)=|\sin y|$ (xx) $f(x,y)=(1+x^2)y$

(xxi) $f(x,y)=g(x)y$ where $g(x)$ is continuous on R.

(xxii) $f(x,y)=(1+g^2(x))y$, where $g(x)$ is continuous on R.

(xxiii) $f(x,y)=a_0(x)y^2+a_1(x)y+a_2(x)$, where a_0, a_1 and a_2 are continuous functions on R.

(xxiv) $f(x,y)=x|y|$

4. Discuss the existence and uniqueness of the following initial value problems. Also find the solution, wherever possible:

(i) $\frac{dy}{dx}=y^{\frac{4}{3}}$, $y(0)=0$ (ii) $\frac{dy}{dx}=y^{\frac{4}{3}}, y(0)=1$ (iii) $\frac{dy}{dx}=y^{\frac{4}{3}}, y(1)=0$

(iv) $\frac{dy}{dx}=y^{\frac{4}{3}}, y(1)=1$ (v) $\frac{dy}{dx}=y^{\frac{2}{3}}, y(0)=0$ (vi) $\frac{dy}{dx}=y^{\frac{2}{3}}, y(0)=1$

(vii) $\frac{dy}{dx}=y^{\frac{2}{3}}, y(1)=0$ (viii) $\frac{dy}{dx}=y^{\frac{2}{3}}, y(1)=1$ (ix) $\frac{dy}{dx}=\sqrt{|y|}, y(0)=0$

(x) $\frac{dy}{dx}=\sqrt{|y|}, y(0)=1$ (xi) $\frac{dy}{dx}=\sqrt{|y|}, y(1)=0$ (xii) $\frac{dy}{dx}=\sqrt{|y|}, y(1)=1$

(xiii) $\frac{dy}{dx}=y^2, y(0)=1$ [NET June 2013] (xiv) $\frac{dy}{dx}=(1+x^2)y, y(0)=1$

(xv) $\frac{dy}{dx}=(1+f^2(x))y, y(0)=1$, where f is continuous on R [NET DEC 2011]

(xvi) $\frac{dy}{dx}=1+y^2, y(0)=0$ (xvii) $\frac{dy}{dx}=e^{2y}, y(0)=0$

(xviii) $\frac{dy}{dx}=xy^{\frac{1}{3}}, y(0)=0$ [NET June 2013] (xix) $\frac{dy}{dx}=\sqrt{y}, y(1)=0$

(xx) $\frac{dy}{dx}=60y^{\frac{2}{5}}, y(0)=0$ [NET Dec. 2012] (xxi) $x\frac{dy}{dx}=3y, y(1)=1$

(xxii) $\frac{dy}{dx}=f(x)y(x), y(0)=1$, where f is a continuous function on R [NET June 2012]

5. Find the largest possible interval for x in which Picard's uniqueness theorem (PUT) guarantees the existence of a unique solution. Also, wherever possible, find the interval for x in which the solution actually exist.

(i) $\frac{dy}{dx}=1+y^2; y(0)=0$ (ii) $\frac{dy}{dx}=e^{2y}, y(0)=0$ (iii) $\frac{dy}{dx}=x^2+y^2, y(0)=0$

(iv) $\frac{dy}{dx}=16+y^2, y(0)=0$ (v) $\frac{dy}{dx}=y, y(0)=1, R:|x|\le 1, |y-1|\le 1$

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(vi) $\frac{dy}{dx}=\frac{1+2x+3y}{2+x^2+y^2},\ y(0)=0,\ \ R:|x|\le 2,|y|\le 1$

(vii) $\frac{dy}{dx}=\frac{1+x+y}{2+x^2+y^2},\ y(1)=1,\ \ R:|x-1|\le 1,|y-1|\le 1$

(viii) $\frac{dy}{dx}=\frac{2xy}{4+x^2+y^2},\ y(-1)=1,\ \ R:|x+1|\le 1,|y-1|\le 1$

(ix) $\frac{dy}{dx}=\frac{3xy}{2+\cos(xy)},\ \ y(0)=0,\ \ R:|x|\le 1,|y|\le 2$

Answers

1. (i) Yes (ii) Yes (iii) Yes (iv) Yes (v) Yes
   (vi) No (vii) No (viii) No (ix) Yes (x) Yes
   (xi) No (xii) Yes (xiii) No (xiv) No (xv) No
   (xvi) No (xvii) No (xviii) Yes (xix) Yes (xx) Yes
   (xxi) Yes (xxii) Yes (xxiii) Yes (xxiv) Yes
2. (i) Yes (ii) Yes (iii) Yes (iv) Yes (v) Yes
   (vi) No (vii) No (viii) No (ix) Yes (x) Yes
   (xi) No (xii) Yes (xiii) No (xiv) No (xv) No
   (xvi) No (xvii) No (xviii) Yes (xix) Yes (xx) Yes
   (xxi) Yes (xxii) Yes (xxiii) Yes (xxiv) Yes
3. (i) Yes (ii) Yes (iii) Yes (iv) Yes (v) Yes
   (vi) Yes (vii) Yes (viii) Yes (ix) Yes (x) Yes
   (xi) Yes (xii) Yes (xiii) Yes (xiv) Yes (xv) Yes
   (xvi) Yes (xvii) Yes (xviii) Yes (xix) Yes (xx) Yes
   (xxi) Yes (xxii) Yes (xxiii) Yes (xxiv) Yes
4. (i) trivial solution , no actual solution, unique solution

   (ii) $y^{\frac{1}{3}}=\frac{3}{3-x}$, non trivial solution, unique solution

   (iii) trivial solution, no actual solution, unique solution

   (iv) $y^{\frac{1}{3}}=\frac{-3y_0^{\frac{1}{3}}}{y_0^{\frac{1}{3}}x-3-x_0y_0^{\frac{2}{3}}}$, non trivial solution, unique solution

(v) trivial solution, $y^{\frac{1}{3}} = \frac{x}{3}$, infinite solution

(vi) $y^{\frac{1}{3}} = \frac{x+3}{3}$, non trivial solution

(vii) trivial solution, $y^{\frac{1}{3}} = \frac{x-1}{3}$, infinite solution

(viii) $y^{\frac{1}{3}} - y_0^{\frac{1}{3}} = \frac{x-x_0}{3}$, non trivial solution

(ix) trivial solution, $y^{\frac{1}{2}} = \frac{x}{2}$, infinite solution

(x) $y^{\frac{1}{2}} = \frac{x+2}{2}$, non trivial solution, unique solution

(xi) trivial solution, $y^{\frac{1}{2}} = \frac{x-1}{2}$, infinite solution

(xii) $y^{\frac{1}{2}} - y_0^{\frac{1}{2}} = \frac{x-x_0}{2}$, non trivial solution

(xiii) $y = \frac{1}{1-x}$, non trivial solution

(xiv) $y = e^{x+\frac{x^3}{3}}$, non trivial solution, unique solution

(xv) $y = e^{x+\int f^2(x)dx}$, non trivial solution

(xvi) $y = \tan x$, non trivial solution, unique solution

(xvii) $y = -\frac{1}{2}\log(1-2x)$, non trivial solution, unique solution

(xviii) $y^{\frac{2}{3}} = \frac{x^2}{3}$, trivial solution, infinite solution

(xix) $y^{\frac{1}{2}} = \frac{x-1}{2}$, trivial solution, infinite solution

(xx) $y^{\frac{3}{5}} = 36x$, trivial solution, infinite solution

(xxi) $y = x^3$, non trivial solution, unique solution

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(xxii) $y = e^{\int f(x)\,dx}$, non trivial solution, unique solution

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Assignment-7

--------------------- **S C Q** ---------------------

1. The initial value problem
$$\left(x^2-x\right)\frac{dy}{dx}=(2x-1)y,\quad y(x_0)=y_0$$
has a unique solution, if (x_0, y_0) equals
   1. (2, 1) 2. (1, 1)
   3. (0, 0) 4. (0, 1)

(GATE 2002)

2. The initial value problem
$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=0;$$
$$y(0)=1, \left(\frac{dy}{dx}\right)_{x=0}=0 \text{ has}$$
   1. a unique solution
   2. no solution
   3. infinitely many solutions
   4. two linearly independent solutions

(GATE 2006)

3. The initial value problem
$$x\frac{dy}{dx}=y+x^2,\ x>0;\ y(0)=0,\ \text{has}$$
   1. infinitely many solutions
   2. exactly two solutions
   3. a unique solution
   4. no solution

(GATE 2011)

4. Let y be a solution of $y'=e^{-y^2}-1$ on [0,1] which satisfies $y(0)=0$. Then,
   1. $y(x)>0$ for $x>0$
   2. $y(x)<0$ for $x>0$
   3. y changes sign in [0, 1]
   4. $y\equiv 0$ for $x>0$

(GATE 2008)

5. Let P be a polynomial of degree N, with $N\geq 2$. Then the initial value problem $u'(t)=P\left(u(t)\right), u(0)=1$ has always
   1. a unique solutions in $\mathbb{R}$.
   2. N number of distinct solution in $\mathbb{R}$.
   3. no solution in any interval containing 0 for some P.
   4. a unique solution in an interval containing 0.

(CSIR NET June 2011)

6. Consider the equation
$$\frac{dy}{dt}=\left(1+f^2(t)\right)y(t), y(0)=1; t\geq 0$$ where f is a bounded continuous function on $[0,\infty)$. Then
   1. This equation admits a unique solution $y(t)$ and further $\lim\limits_{t\to\infty} y(t)$ exists and is finite
   2. This equation admits two linearly independent solutions
   3. This equation admits a bounded solution for which $\lim\limits_{t\to\infty} y(t)$ does not exist
   4. This equation admits a unique solution $y(t)$ and further $\lim\limits_{t\to\infty} y(t)=\infty$

(CSIR NET Dec 2011)

7. Consider the initial value problem
$$y'(t)=f(t)y(t),\ y(0)=1 \text{ where } f:\mathbb{R}\to\mathbb{R}$$
is continuous. Then this initial value problem has
   1. infinitely many solutions for some f.
   2. a unique solution in $\mathbb{R}$.
   3. no solution in $\mathbb{R}$ for some f.
   4. a solution in an interval containing 0, but not on $\mathbb{R}$ for some f.

(CSIR NET June 2012)

8. Let $y_1(x)$ and $y_2(x)$ be the solutions of the differential equation $\frac{dy}{dx}=y+17$ with initial conditions $y_1(0)=0,\ y_2(0)=1.$ Then
   1. y_1 and y_2 will never intersect.
   2. y_1 and y_2 will never intersect at x =17.
   3. y_1 and y_2 will intersect at $x = e$.
   4. y_1 and y_2 will intersect at $x = 1$.

(CSIR NET Dec 2012)

9. Consider the initial value problem (IVP) $\frac{dy}{dx}=y^2, y(0)=1, (x,y)\in\mathbb{R}\times\mathbb{R}.$ Then there exists a unique solution of the IVP on]
   1. $(-\infty,\infty)$
   2. $(-\infty,1)$
   3. $(-2, 2)$
   4. $(-1,\infty)$

(CSIR NET June 2013)

10. Consider the initial value problem in $\mathbb{R}^2$.
$Y'(t)=AY+BY;\ Y(0)=Y_0,$ where
$A=\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}, B=\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}.$ Then $Y(t)$ is given by
    1. $e^{tA}e^{tB}Y_0$
    2. $e^{tB}e^{tA}Y_0$
    3. $e^{t(A+B)}Y_0$
    4. $e^{-t(A+B)}Y_0$

(CSIR NET June 2014)

11. Let $y:\mathbb{R}\to\mathbb{R}$ be differentiable and satisfy the ODE : $\left.\begin{matrix}\frac{dy}{dx}=f(y), x\in\mathbb{R}\\ y(0)=y(1)=0\end{matrix}\right\}$ where $f:\mathbb{R}\to\mathbb{R}$ is a Lipschitz continuous function. Then
    1. $y(x)=0$ iff $x\in\{0,1\}$
    2. y is bounded
    3. y is strictly increasing
    4. $\frac{dy}{dx}$ is unbounded

(CSIR NET Dec 2014)

12. The initial value problem $y'=2\sqrt{y},\ y(0)=a$, has
    1. a unique solution if $a<0$
    2. no solution if $a>0$
    3. infinitely many solutions if $a=0$
    4. a unique solution if $a\geq 0$

(CSIR NET June 2015)

13. Consider the ordinary differential equation $y'=y(y-1)(y-2)$. Which of the following statements is true ?
    1. If $y(0)=0.5$ then y is decreasing
    2. If $y(0)=1.2$ then y is increasing
    3. If $y(0)=2.5$ then y is unbounded
    4. If $y(0)<0$ then y is bounded below

(CSIR NET June 2018)

---------------------- **M C Q** --------------------

1. The differential equation $\frac{dy}{dx}=60\left(y^2\right)^{\frac{1}{5}};\ x>0$
$y(0)=0$ has
   1. a unique solution.
   2. two solutions.
   3. no solution.
   4. infinite number of solutions.

(CSIR NET Dec 2012)

2. Consider the initial value problem $(IVP)\frac{dy}{dx}=xy^{\frac{1}{3}}, y(0)=0, (x,y)\in\mathbb{R}\times\mathbb{R}.$
Then, which of the following are correct ?
   1. The function $f(x,y)=xy^{\frac{1}{3}}$ does not satisfy a Lipschitz condition with respect to y in any neighbourhood of y=0.
   2. There exists a unique solution for the IVP.
   3. There exists no solution for the IVP.
   4. There exist more than one solution for the IVP.

(CSIR NET June 2013)

3. Let $y:\mathbb{R}\to\mathbb{R}$ satisfy the initial value problem $y'(t)=1-y^2(t), t\in\mathbb{R},\ y(0)=0.$ Then

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1. $y(t_1)=1$ for some $t_1 \in \mathbb{R}$.
2. $y(t) > -1$ for all $t \in \mathbb{R}$.
3. y is strictly increasing in $\mathbb{R}$.
4. y is increasing in (0, 1) and decreasing in $(1,\infty)$.

(CSIR NET Dec 2013)

4. Let $u(t)$ be a continuously differentiable function taking non negative values for $t>0$ satisfying $u'(t)=3u(t)^{\frac{2}{3}}$ and $u(0)=0$. Which of the following are possible solutions of the above equations ?
   1. $u(t)=0$
   2. $u(t)=t^3$
   3. $u(t)=\begin{cases} 0 \text{ for } 0<t<1 \\ (t-1)^3 \text{ for } t\ge 1 \end{cases}$
   4. $u(t)=\begin{cases} 0 \text{ for } 0<t<3 \\ (t-3)^3 \text{ for } t\ge 3 \end{cases}$

(CSIR NET June 2014)

5. Let $u(t)$ be a continuously differentiable function taking nonnegative values for $t>0$ and satisfying $u'(t)=4u^{\frac{3}{4}}(t);\ u(0)=0$. Then
   1. $u(t)=0$
   2. $u(t)=t^4$
   3. $u(t)=\begin{cases} 0 & \text{for } 0<t<1 \\ (t-1)^4 & \text{for } t\ge 1 \end{cases}$
   4. $u(t)=\begin{cases} 0 & \text{for } 0<t<10 \\ (t-10)^4 & \text{for } t\ge 10 \end{cases}$

(CSIR NET Dec 2015)

6. For $\lambda \in \mathbb{R}$, consider the differential equation $y'(x)=\lambda \sin(x+y(x)), y(0)=1$. Then this initial value problem has :
   1. no solution in any neighbourhood of 0
   2. a solution in $\mathbb{R}$ if $|\lambda|<1$.
   3. a solution in a neighbourhood of 0
   4. a solution in $\mathbb{R}$ only if $|\lambda|>1$

(CSIR NET June 2016)

7. Consider the initial value problem $y'(t)=f(y(t)),\quad y(0)=a\in\mathbb{R}$ where $f:\mathbb{R}\to\mathbb{R}$. Which of the following statements are necessarily true ?
   1. There exists a continuous function $f:\mathbb{R}\to\mathbb{R}$ and $a\in\mathbb{R}$ such that the above problem does not have a solution in any neighbourhood of 0.
   2. The problem has a unique solution for every $a\in\mathbb{R}$ when f is Lipschitz continuous.
   3. When f is twice continuously differentiable, the maximal interval of existence for the above initial value problem is $\mathbb{R}$.
   4. The maximal interval of existence for the above problem is $\mathbb{R}$ when f is bounded and continuously differentiable.

(CSIR NET Dec 2016)

8. Assume that $a:[0,\infty)\to\mathbb{R}$ is a continuous function. Consider the ordinary differential equation

   $y'(x)=a(x)y(x),\ x>0,\ y(0)=y_0\neq 0$.

   Which of the following statements are true ?
   1. If $\int_0^\infty |a(x)|dx<\infty$, then y is bounded
   2. If $\int_0^\infty |a(x)|dx<\infty$, then $\lim_{x\to\infty} y(x)$ exists
   3. If $\lim_{x\to\infty} a(x)=1$, then $\lim_{x\to\infty}|y(x)|=\infty$

4. If $\lim_{x\to\infty} a(x)=1$, then y is montone

(CSIR NET June 2018)

Answers

----------------------- S C Q --------------------

1. 1	2. 2	3. 1
4. 4	5. 4	6. 4
7. 2	8. 1	9. 2
10. 3	11. 2	12. 3
13. 3		

---------------------- M C Q -------------------

1. 4	2. 1,4	3. 2,3
4. 1,2,3,4	5. 1,2,3,4	6. 2,3
7. 2,4	8. 1,2,3,4	

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Chapter - 8

Equations of First Order and Higher Degree and singular solution

Introduction : In this chapter, we shall study those differential equations, which are of first order but not of first degree. The general form of such differential equation is :

$$P_0\left(\frac{dy}{dx}\right)^n + P_1\left(\frac{dy}{dx}\right)^{n-1} + P_2\left(\frac{dy}{dx}\right)^{n-2} + \dots + P_{n-1}\frac{dy}{dx} + P_n = 0 \qquad \text{......(1)}$$

where $P_0, P_1, P_2, ..., P_{n-1}, P_n$ are functions of x and y.

Usually , we denote $\frac{dy}{dx}$ by p, so that equation (1) takes the form

$$P_0\, p^n + P_1 p^{n-1} + P_2\, p^{n-2} + \dots + P_{n-1}p + P_n = 0$$

We shall solve such equations and find the general and singular solutions of these equations.

The differential equation of first order but not of first degree can also be expressed as

$$f(x, y, p) = 0, \ \text{ where } \ p = \frac{dy}{dx}.$$

To solve such differential equations, we shall consider the following five cases.

(i) Equations solvable for p

(ii) Equations solvable for x

(iii) Equations solvable for y

(iv) Lagrange's equation and Clairaut's equation.

(v) Equations reducible to Clairaut's equation .

We shall also find the singular solution of such equation. Let us discuss these cases one by one.

1. Equations solvable for *p* :

A differential equation of first order and nth degree is of the form

$$P_0\, p^n + P_1\, p^{n-1} + P_2 p^{n-2} + \dots + P_{n-1}p + P_n = 0 \qquad \text{......(1)}$$

where $P_0, P_1, P_2, ..., P_{n-1}, P_n$ are functions of x and y. Let left hand side of (1) be solvable for p, then it is resolved in to n linear factors of the type

$$[p - f_1(x, y)][p - f_2(x, y)].......[p - f_n(x, y)] = 0$$

$$\Rightarrow \qquad p = f_1(x, y),\ p = f_2(x, y), ..., p = f_n(x, y)$$

$$\text{or} \qquad \frac{dy}{dx} = f_1(x, y)\ ,\ \frac{dy}{dx} = f_2(x, y)\ ,\ ....,\ \frac{dy}{dx} = f_n(x, y)$$

These are differential equations of first order and first degree and can be solved easily. Let the solutions of these equations be

$$F_1(x,y,c_1)=0,\ F_2(x,y,c_2)=0,\ldots\ldots\ldots\ldots,F_n(x,y,c_n)=0$$

The general solution of given differential equation is given by

$$F_1(x,y,c_1)\ F_2(x,y,c_2)\ \ \ldots\ldots.\ F_n(x,y,c_n)=0$$

Since the given differential equation is of first order ; therefore , the general solution can not have more than one arbitrary constant , so we take

$$c_1=c_2=c_3\ldots\ldots\ldots\ldots.c_n=c \text{ (say)}$$

Hence the general solution of the given equation can be put as

$$F_1(x,y,c)\ F_2(x,y,c)\ \ \ldots\ldots.\ F_n(x,y,c)=0$$

Working Rule :

1. Reduce the given differential equation in linear factors of p
2. Solve individually these linear factors of p
3. Multiply all the solutions obtained in step 2 and equate it to zero. This gives the general solution of the given differential equation.

Example 1 : Solve $p^2 - 7p + 12 = 0$

Solution : The given differential equation is

$$p^2 - 7p + 12 = 0. \qquad \ldots\ldots(1)$$

or $(p-3)(p-4)=0$

$\Rightarrow$ $p = 3$ or $p = 4$

i.e. $\dfrac{dy}{dx} = 3$ or $\dfrac{dy}{dx} = 4$

$\Rightarrow$ $dy = 3\,dx$ | $dy = 4\,dx$

Integrating , $y = 3x + c$ Integrating , $y = 4x + c$

or $y - 3x - c = 0$ or $y - 4x - c = 0$

Therefore the general solution of given equation is $(y-3x-c)(y-4x-c)=0.$

2. Equations Solvable for x :

A differential equation solvable for x can be put in the form

$$x = f\left(y,p\right) \qquad \ldots\ldots.(1)$$

Differentiating w. r. t. y , we get

$$\frac{dx}{dy} = g\left(y, p, \frac{dp}{dy}\right)$$

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$\Rightarrow$ $$\frac{1}{p} = g\left(y, p, \frac{dp}{dy}\right) \quad ....(2)$$

This equation involves two variables *y* and *p*. Let solution of this equation be

$$\phi(y, p, c) = 0 \quad ....(3)$$

Elimination of *p* from (1) and (3) gives the required solution. If elimination of *p* is not possible, then expressions of *x* and *y* in terms of *p* together form the solution of the given equation.

Working Rule :

1. Express *x* explicitly in terms of *p* and *y*.

2. Differentiate the equation w.r.t. *y* and replace $\frac{dx}{dy}$ by $\frac{1}{p}$. The equation obtained involves *p* and *y*.

3. Solve the equation obtained in step 2 and eliminate *p* from this solution and the given equation.

Example 2 : Solve $y - 2px = y^2p^3$

Solution : The given differential equation is $y = 2px + y^2p^3 \quad ......(1)$

$\Rightarrow$ $$x = \frac{y}{2p} - \frac{y^2p^2}{2}$$

Differentiating w.r.t. *y*, we have

$$\frac{dx}{dy} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - yp^2 - py^2\frac{dp}{dy}$$

or $$\frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - yp^2 - py^2\frac{dp}{dy}$$

or $$\frac{1}{2p} + yp^2 = -\left(\frac{y}{2p^2} + py^2\right)\frac{dp}{dy}$$

or $$\frac{1}{2p} + yp^2 = -\frac{y}{p}\left(\frac{1}{2p} + p^2y\right)\frac{dp}{dy}$$

or $$\left(\frac{1}{2p} + yp^2\right) + \frac{y}{p}\left(\frac{1}{2p} + yp^2\right)\frac{dp}{dy} = 0$$

or $$\left(\frac{1}{2p} + yp^2\right)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$$

or $$1 + \frac{y}{p}\frac{dp}{dy} = 0$$

or $$\frac{dy}{y}+\frac{dp}{p}=0$$

Integrating , we get $$\log y+\log p=\log c$$

or $$\log yp=\log c$$

i.e. $$yp=c \quad \Rightarrow \quad p=\frac{c}{y}$$

Thus from equation (1), we have

$$y=\frac{2c}{y}x+y^2\frac{c^3}{y^3}$$

or $$y^2=2cx+c^3$$

which is the general solution of given equation.

3. Equations Solvable for y :

A differential equation solvable for y can be put in the form

$$y=f(x,p) \qquad ....(1)$$

Differentiating w.r.t. y , we get

$$\frac{dy}{dx}=g\left(x,p,\frac{dp}{dx}\right)$$

$$\Rightarrow \qquad p=g\left(x,p,\frac{dp}{dx}\right) \qquad ....(2)$$

This equation involves two variables x and p. Let solution of this equation be

$$\phi(x,p,c)=0 \qquad ....(3)$$

Elimination of p from (1) and (3) gives the required solution. If elimination of p is not possible, then expressions of x and y in terms of p together form the solution of the given equation.

Working Rule :

1. Express y explicitly in terms of p and x.
2. Differentiate the equation w.r.t. x and replace $\frac{dy}{dx}$ by p. The obtained equation involves p and x.
3. Solve the equation obtained in step 2 and eliminate p from this solution and the given equation.

Example 3 : Solve $y = 3x + \log p$

Solution : The given differential equation is

$$y = 3x + \log p \qquad ......(1)$$

Differentiating w.r.t. x, we have

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$$\frac{dy}{dx} = 3 + \frac{1}{p}\frac{dp}{dx}$$

or $$p = 3 + \frac{1}{p}\frac{dp}{dx}$$

or $$p - 3 = \frac{1}{p}\frac{dp}{dx}$$

or $$dx = \frac{dp}{p(p-3)}$$

Integrating both sides, we have

$$x = \frac{1}{3}\int\left(-\frac{1}{p} + \frac{1}{p-3}\right)dp + c' \qquad \text{[Making partial fractions]}$$

or $$x = \frac{1}{3}\left[-\log p + \log(p-3)\right] + c'$$

or $$3x = -\log\frac{p}{p-3} + 3c'$$

or $$3x = -\log\frac{p}{p-3} + \log c'' \qquad [3c' = \log c'']$$

or $$\log e^{3x} = \log c''\left(\frac{p-3}{p}\right) \qquad [x = \log e^{x}]$$

or $$e^{3x} = c''\frac{p-3}{p}$$

or $$e^{3x} = c''\left(1 - \frac{3}{p}\right)$$

or $$1 - ce^{3x} = \frac{3}{p} \quad \text{or} \quad p = \frac{3}{1 - ce^{3x}} \qquad \left[\because \frac{1}{c''} = c\right]$$

Then from equation (1), we have $y = 3x + \log\dfrac{3}{1 - ce^{3x}}$ which is the required solution.

Exercise 8.1

Solve the following differential equations :

1. $p^2 - 5p + 6 = 0$

2. $x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0$

3. $y = 2px + p^2 y$

4. $x = y + p^2$

5. $y + px = p^2 x^4$

6. $y = p\sin p + \cos p$

Answers

1. $(y - 3x - c)(y - 2x - c) = 0.$

2. $(x^3 y - c)(y - cx^2) = 0.$

3. $y^2 = 2cx + c^2$

4. $x = c - [2p + 2\log(p-1)],\ y = c - [p^2 + 2p + 2\log(p-1)].$

5. $y = -\frac{c}{x} + c^2$

6. $x = c + \sin p$ with given relation.

4. Lagrange's Equation :

The differential equation of the form

$y = x\phi(p) + \psi(p)$ is known as Lagrange's equation

e.g. $\quad y = p^2 x + \sin p$

To find the solution of Lagrange's equation :

The given differential equation is

$$y = x\phi(p) + \psi(p) \qquad ....(1)$$

Differentiating w. r. t. x, we have

$$\frac{dy}{dx} = \phi(p) + x\phi'(p)\frac{dp}{dx} + \psi'(p)\frac{dp}{dx}$$

$$\Rightarrow \quad p = \phi(p) + x\phi'(p)\frac{dp}{dx} + \psi'(p)\frac{dp}{dx}$$

$$\Rightarrow \quad p - \phi(p) = \left[x\phi'(p) + \psi'(p)\right]\frac{dp}{dx}$$

$$\Rightarrow \quad \left[p - \phi(p)\right]\frac{dx}{dp} = x\phi'(p) + \psi'(p)$$

$$\Rightarrow \quad \frac{dx}{dp} = \frac{\phi'(p)}{p - \phi(p)}x + \frac{\psi'(p)}{p - \phi(p)}$$

$$\Rightarrow \quad \frac{dx}{dp} - \frac{\phi'(p)}{p - \phi(p)}x = \frac{\psi'(p)}{p - \phi(p)}$$

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This is a linear differential equation in terms of x and p. Let the solution of this equation be

$$f\left(x, p, c\right) = 0 \qquad ....(2)$$

Then (1) and (2) taken together provide the required solution.

Working rule : The given differential equation is of the form

$$y = x\phi\left(p\right) + \psi\left(p\right)$$

Differentiate this equation w.r.t. x and replace $\frac{dy}{dx}$ by p. We get a linear differential equation in p and x. Solve it by the method of integrating factor. This solution taken together with the given equation will provide the required solution.

Example 4 : Solve $y = xp^2 + p$.

Solution : The given differential equation is $y = xp^2 + p$ ......(1)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = p^2 + x.2p\frac{dp}{dx} + \frac{dp}{dx}$$

or
$$p = p^2 + (2px+1)\frac{dp}{dx}$$

or
$$p - p^2 = (2px+1)\frac{dp}{dx}$$

or
$$\frac{dx}{dp} = \frac{2p}{p-p^2}x + \frac{1}{p-p^2}$$

or
$$\frac{dx}{dp} + \frac{2}{p-1}x = \frac{1}{p(1-p)} \qquad ......(2)$$

which is a linear differential equation.

Here
$$I.F. = e^{2\int \frac{1}{p-1}dp} = e^{\log(p-1)^2} = (p-1)^2$$

$\therefore$ The solution is
$$x.(p-1)^2 = \int (p-1)^2 \frac{1}{p(1-p)} dp + c$$

$$(p-1)^2 x = -\int \frac{p-1}{p} dp + c$$

$$(p-1)^2 x = -p + \log p + c$$

or $$x = \frac{-p + \log p + c}{(p-1)^2} \quad \text{......(3)}$$

Putting the value of x in (1), we have

$$y = \frac{-p^3 + p^2 \log p + cp^2}{(p-1)^2} + p \quad \text{......(4)}$$

Equation (3) and equation (4) taken together provide the required solution.

5. Clairaut's equation (a particular case of Lagrange's equation) :

In the Lagrange's equation

$$y = x\phi(p) + \psi(p)$$

If we take $\phi(p) = p$, then we get

$$y = xp + \psi(p)$$

This is called Clairaut's equation.

To solve Clairaut's equation :

The given equation is of the type $y = px + f(p)$ ......(1)

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\Rightarrow \quad p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\Rightarrow \quad \left[x + f'(p)\right]\frac{dp}{dx} = 0$$

$$\Rightarrow \quad \frac{dp}{dx} = 0 \qquad \text{[Neglecting } x + f'(p) = 0\text{]}$$

Now $\frac{dp}{dx} = 0 \quad \Rightarrow \quad p = c$

Putting $p = c$ in (1), we get the general solution of (1) as $y = cx + f(c)$

Thus, solution of Clairaut's equation is obtained by just replacing p by c.

Example 5 : Solve $\sin px \cos y = \cos px \sin y + p$

Solution : The given equation is $\sin px \cos y = \cos px \sin y + p$ ......(1)

$$\Rightarrow \quad \sin(p\,x - y) = p$$

$$\Rightarrow \quad px - y = \sin^{-1} p$$

$$\Rightarrow \quad y = p\,x - \sin^{-1} p$$

which is Clairaut's equation.

$\therefore$ The solution is given by $y = cx - \sin^{-1} c$.

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6. Equations reducible to Clairaut's form :

Some differential equations of first order and higher degree can be reduced to Clairaut's form by making suitable substitutions.

There is no general method to find the proper substitutions. These can be learned by practice only.

Example 6 : Solve $e^{5x}(p-1)+p^5e^{4y}=0$.

Solution : The given differential equation is

$$e^{5x}(p-1)+p^5e^{4y}=0 \qquad \text{......(1)}$$

Put $X=e^x$ and $Y=e^y$

$\Rightarrow$ $dX=e^x dx$ and $dY=e^y\,dy$

$$\Rightarrow \quad \frac{dY}{dX}=\frac{e^y dy}{e^x dx}=\frac{e^y}{e^x}\frac{dy}{dx}=\frac{e^y}{e^x}.p$$

Assume
$$P=\frac{dY}{dX}=\frac{e^y}{e^x}.p=\frac{Y}{X}.p$$

$$\Rightarrow \quad p=\frac{X}{Y}P$$

Thus from equation (1), we have

$$X^5\left[\left(\frac{X}{Y}P\right)-1\right]+\frac{X^5}{Y^5}P^5Y^4=0$$

or $$\frac{X}{Y}P-1+\frac{P^5}{Y}=0$$

or $$PX-Y+P^5=0$$

or $$Y=PX+P^5$$

which is a clairaut's equation. Therefore the solution is given by

$$Y=cX+c^5 \qquad \text{[By replacing } P \text{ by } c\text{]}$$

or $$e^y=ce^x+c^5$$

Exercise 8.2

Solve the following differential equations :

1. $y=(1+p)x+p^2$
2. $y=xp+\dfrac{a}{p}$
3. $(px-y)(x+py)=a^2p$
4. $(x^2+y^2)(1+p)^2-2(x+y)(1+p)(x+yp)+(x+yp)^2=0$

Answers

1. $y = 2 - p^2 + ce^{-p}(1+p)$ and together with given relation. 2. $y = cx + \dfrac{a}{c}$

3. $y^2 = cx^2 - \dfrac{a^2c}{c+1}$ 4. $x^2 + y^2 = c(x + y) - \dfrac{c^2}{4}$

Singular solution : A singular solution in this stronger sense is often given as tangent to every solution from a family of solutions. A singular solution is the envelope of the family of solutions cannot be obtained from general solution by giving particular values of arbitrary constants.

The singular solution is related to the general solution by its being what is called the envelope of that family of curves representing the general solution. An envelope is defined as the curve that is tangent to a given family of curves.

Envelope : A curve which touches each member of a one-parameter family of curves and at each point is touched by some member of the family, is called the envelope of the that parameter family of curves.

Example 1 : $y' = 4$ has the general solution $y = (x+c)^2$, which is a family of parabolas. The line $y(x) = 0$ is also a solution of the differential equation, but it is not a member of the family constituting the general solution. Hence line $y(x) = 0$ is singular solution of the differential equation.

Example 2 : Consider family of circles with centre lying on *x*-axis and radius 3 is. This family is given by $(x-a)^2 + y^2 = 9$. And corresponding differential equation can be obtained by eliminating a parameter a.

$\Rightarrow \quad (x-a) = -yy' \qquad \Rightarrow \quad (yy')^2 + y^2 = 9$

$\Rightarrow \quad y^2\left[y'^2 + 1\right] = 9$

Hence $(x-a)^2 + y^2 = 9$ is general solution of the differential equation $y^2\left[y'^2+1\right] = 9$

also we can easily verify that $y(x) = 3$ and $y(x) = -3$ are the solution of $y^2\left[y'^2+1\right] = 9$ which cannot be obtained from general solution. These both are envelope to the general solution. Hence $y(x) = 3$ and $y(x) = -3$ are singular solution of the differential equation.

Def. Singular Solution : A singular solution is a solution of the differential equation which does not contain any arbitrary constant but can not be derived from general solution.

In other words, it is not obtained just by providing some particular values to arbitrary constants in general solution.

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Def. *p*-discriminant relation of a differential equation :

Let the given equation be $f(x, y, p) = 0$. Let p be treated as a parameter.

The relation obtained by eliminating p between $f(x, y, p) = 0$ and $\frac{\partial f}{\partial p} = 0$ is called p- discriminant relation of the given differential equation.

Def. *c*-discriminant relation of a differential equation :

Let the given equation be $f(x, y, p) = 0$ and let general solution of this equation be $\phi(x, y, c) = 0$.

Let the arbitrary constant c be treated as a parameter.

The relation obtained by eliminating c between $\phi(x, y, c) = 0$ and $\frac{\partial \phi}{\partial c} = 0$ is called c-discriminant relation of the given differential equation.

8. Methods to find the singular Solution :

Method I : Let

$$f(x, y, p) = 0 \qquad \text{......(1)}$$

be the given differential equation.

Differentiating (1) partially w.r.t 'p', we get

$$\frac{\partial f}{\partial p} = 0 \qquad \text{......(2)}$$

Eliminating p between (1) and (2), we get p-discriminant relation, let it be

$$F(x, y) = 0 \qquad \text{......(3)}$$

If (3) satisfies the given equation, it is the required singular solution. If (3) does not satisfy (1), then resolve $F(x, y)$ into simpler factors.

Those factors which satisfy the equation (1), constitute the singular solution.

Method II : Let

$$f(x, y, p) = 0 \qquad \text{......(1)}$$

be the given differential equation and general solution of this equation be

$$\phi(x, y, c) = 0 \qquad \text{......(2)}$$

Differentiating (2) partially w. r. t. 'c', we get

$$\frac{\partial \phi}{\partial c} = 0 \qquad \text{......(3)}$$

Eliminating c between (2) and (3), we get c-discriminant relation, let it be

$$F(x, y) = 0 \qquad \text{......(4)}$$

If (4) satisfies the given equation, it is the required singular solution. If (4) does not satisfy (1), then resolve it into simple factors.

Those factors which satisfy the equation (1) constitute the singular solution.

Remark : (1) If the given equation is quadratic in p, say

$$Ap^2 + Bp + C = 0$$

then p-discriminant can be directly taken as

$$B^2 - 4AC = 0$$

(2) Similarly, if the general solution is quadratic in c, say

$$Ac^2 + Bc + C = 0$$

then c-discriminant can be directly taken as

$$B^2 - 4AC = 0$$

Example 3 : Solve the differential equation $y = px + \frac{a}{p}$ and obtain its singular solution.

Solution : The given differential equation is $y = px + \frac{a}{p}$ ....(1)

which is Clairaut's equation.

Thus its general solution is

$$y = cx + \frac{a}{c} \qquad \text{[Replacing } p \text{ by } c\text{]}$$

$$\Rightarrow \qquad c^2x - cy + a = 0$$

which is quadratic in c.

Hence c-discriminant is given by

$$y^2 - 4ax = 0$$

$$\Rightarrow \qquad y^2 = 4\,ax$$

This is the required singular solution.

(It can be checked that it satisfies the given differential equation)

Exercise 8.3

Find the general and singular solution of the following differential equations :

1. $4xp^2 - (3x-a)^2 = 0$
2. $y^2 - 2pxy + p^2(x^2 - 1) = m^2$
3. $p^2(x^2 - a^2) - 2pxy + y^2 - b^2 = 0$
4. $xp^2 - 2yp + x + 2y = 0$

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Answers

1. $(y-c)^2 = x(x-a)^2$; $x=0$ is singular solution
2. $(y-cx)^2 = m^2+c^2, y^2+m^2x^2=m^2$ is singular solution
3. $(y-cx)^2 = a^2c^2+b^2,\ \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is singular solution.
4. $y-x = x^2c+\dfrac{1}{2c};\ y-x=\pm\sqrt{2}x$ is singular solution.

Analytic function : If a function $f(x)$ is single valued in a domain *D*, then its derivative $f'(a)$ exists if the limit given by $f'(a)=\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ exists and is independent of the path along which $h\to 0$. This function $f(x)$ is said to be analytic at $x=a$ if there exists a neighbourhood $|x-a|<\varepsilon,\ \varepsilon>0,$ at all points of which $f'(x)$ exists. If $f'(x)$ exists at every point in the domain *D* except some exceptional points then $f(x)$ is said to be analytic in *D*. These exceptional points are called singular points or singularities of the function $f(x)$.

Ordinary and singular points : Let us consider a differential equation of the form $y''+P(x)y'+Q(x)y=0$ then a point $x=x_0$ is called an ordinary point of the equation if the functions $P(x)$ and $Q(x)$ both are analytic at point $x=x_0$.

If the point $x=x_0$ is not ordinary point of the equation then this is the singular point.

The singular points are two types :

(a) Regular singular points (b) Irregular singular points

A singular point $x=x_0$ of the given differential equation is known as regular singular point if $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ are analytic at $x=x_0$ otherwise it is called irregular singular point.

Example 1. $(x^2-1)y''+xy'-y=0$

Solution : $(x^2-1)y''+xy'-y=0$

Convert into general form $y''+\dfrac{x}{(x^2-1)}y'-\dfrac{y}{(x^2-1)}=0$

$P(x)=\frac{x}{(x-1)(x+1)}, Q(x)=-\frac{1}{(x-1)(x+1)}$

Now both $P(x)$ and $Q(x)$ are analytic at $x=0$ then $x=0$ is the ordinary point. The function $P(x)$ and $Q(x)$ are not defined at $x=1$ so they are not analytic at $x=1$.

Then $x=1$ is the singular point. Also $(x-1)P(x)=\frac{x}{(x+1)}$ and $(x-1)^2 Q(x)=\frac{x-1}{(x+1)}$ showing both $(x-1)P(x)$ and $(x^2-1)Q(x)$ are analytic at $x=1$ is a regular singular point.

Similarly for $x=-1$ we have to check.

Example 2. $2x^2y''+7x(x+1)y'-3y=0$

Solution : $2x^2y''+7x(x+1)y'-3y=0$

Convert into general form $y''+\frac{7(x+1)}{2x}-\frac{3}{2x^2}y=0$

$P(x)=\frac{7(x+1)}{2x}, Q(x)=\frac{-3}{2x^2}$

We can see that both $P(x)$ and $Q(x)$ are not analytic at $x=0$ so $x=0$ is a singular point. Also $(x-0)P(x)=\frac{7(x+1)}{2}$ and $(x-0)^2 Q(x)=\frac{-3}{2}$ which shows that both $(x-0)P(x)$ and $(x-0)^2 Q(x)$ are analytic at $x=0$. Then $x=0$ is the regular singular point of the given differential equation.

Exercise 8.4

1. (i) $x^3(x-1)y''-2(x-1)y'+3xy=0$ (ii) $(3x+1)xy''-(x+1)y'+2y=0$

   (iii) $(1-x^2)y''-2xy'+x(x+1)y=0$ (iv) $x^2y''+(2-x)y'=0$

2. (i) $y''+(\sin x)y=0$ (ii) $x^2y''+(\sin x)y=0$

   (iii) $x^3y''+(\cos 2x-1)y'+2xy=0$

Answers

1. (i) $x=0;$ irregular, $x=1,$ regular (ii) $x=0,\frac{-1}{3}$ regular

   (iii) $x=1,-1,$ regular (iv) $x=0,$ irregular

2. (i) ordinary point

   (ii) regular point

   (iii) regular singular point

Assignment-8

--------------------- **S C Q** ---------------------

1. The singular solution of $P = \log(Px - y)$ is
   1. $y = x\log x - x$
   2. $y = x\log x + x$
   3. $y = \log x - x$
   4. $y = \log x + x$

2. The singular solution of $y = Px - 2P^2$ is
   1. $x^2 - 8y = 0$
   2. $x^2 + 8y = 0$
   3. $y^2 - 8x = 0$
   4. $y^2 + 8x = 0$

3. The singular solution of $3y = 2Px\frac{-2P^2}{x}$ is
   1. $x^3 + 6y = 0$
   2. $x^3 - 6y = 0$
   3. $x^2 - 6y = 0$
   4. $x^2 - 6 = 0$

4. The singular solution of $xP^2 - 2Py + x + 2y = 0$ is
   1. $(y+x)^2 - 2x^2 = 0$
   2. $(y+x)^2 + 2x^2 = 0$
   3. $(y-x)^2 - 2x^2 = 0$
   4. $(y-x)^2 + 2x^2 = 0$

5. The singular solution of $4P^2(x-2) = 1$ is
   1. $x + 2 = 0$
   2. $y + 2 = 0$
   3. $x + 4 = 0$
   4. $x - 2 = 0$

6. Consider the differential equation $(x-1)y'' + xy' + \frac{1}{x}y = 0$. Then
   1. $x = 1$ is the only singular point
   2. $x = 0$ is the only singular point
   3. both $x = 0$ and $x = 1$ are singular points
   4. neither $x = 0$ nor $x = 1$ are singular points

(CSIR NET Dec 2017)

Answers

--------------------- **S C Q** ---------------------

1. 1	2. 1	3. 2	4. 3
5. 4	6. 3		

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Integral Equation : An integral equation is the equation in which the unknown function $u(x)$ appears inside an integral sign. The most standard type of integral equation in $u(x)$ is of the form

$$u(x)=f(x)+\lambda\int_{a(x)}^{b(x)}K(x,\xi)u(\xi)d\xi$$

where $a(x)$ and $b(x)$ are the limits of integration, λ is a constant parameter and $K(x,t)$ is known function of two variables x and ξ called kernel or the nucleus of the integral equation. The unknown function $u(x)$ that will be determined appears inside the integral sign. In many other cases, the unknown function $u(x)$ appears inside and out side the integral sign. The function $f(x)$ and $K(x,\xi)$ are given in advance. It is to be noted that the limit of integration $a(x)$ and $b(x)$ may be both variables, constants or mixed.

Fredholm integral equation : If the limits of integration are fixed, (i.e., if the domain of integration is fixed), integral equation is called a Fredholm integral equation given in the form :

$u(x)=f(x)+\lambda\int_{a}^{b}K(x,\xi)u(\xi)d\xi$ where a and b are constants.

Volterra integral equation : An integral equation is said to be volterra integral equation is the upper limit of integration is a variable $u(x)=f(x)+\lambda\int_{a}^{x}K(x,\xi)u(\xi)d\xi$

General form of an Integral Equation: $h(x)u(x)=f(x)+\lambda\int_{a(x)}^{b(x)}K(x,\xi)u(\xi)d\xi$

Classification :

(a) Volterra / Fredholm **(b)** Homogeneous / Non-homogeneous **(c)** First kind / Second kind

Volterra Integral equation : If upper limit is variable, then equation is Volterra.

Fredholm Integral equation : If upper limit is constant, then equation is Fredholm.

Homogeneous equation : If in general equation $f(x)=0$, it is known as Homogeneous equation.

Non-Homogeneous equation : If in general equation $f(x)\neq 0$, it is known as Non-Homogeneous equation.

First kind : If $h(x)=0$. **Second kind :** If $h(x)\neq 0$.

Example :

(i) $u(x)=\cos x+3\int_0^x (x+\xi)\ u(\xi)d\xi$, Non-Homogeneous Volterra Integral equation of second kind.

(ii) $u(x)=\cos x+3\int_0^1 (x+t)\ u(t)dt$, Non-Homogeneous Fredholm Integral equation of second kind.

(iii) $1=\cos x+3\int_0^x (x+t)\ u(t)dt$, Non-Homogeneous Volterra Integral equation of first kind.

Write down the type of each of the following integral equations and also verify that the given functions are solution of the corresponding integral equation.

Integro-differential equation : An equation that includes both integrals and derivatives of the unknown function $u(x)$ is called integro-differential equation.

Fredholm integro-differential equation is of the form

$$u^k(x)=f(x)+\lambda\int_a^b K(x,\xi)u(\xi)d\xi$$

Volterra integro-differential equation is of the form

$$u^k(x)=f(x)+\lambda\int_a^x K(x,\xi)u(\xi)d\xi ,$$

where $u^k(x)=\dfrac{d^k u}{dx^k}$

Example 1. $x^3=\int_0^x (x-\xi)^2 u(\xi)d\xi$; $u(x)=3$

Solution : Volterra non-homogeneous integral equation of First kind

R.H.S. : $\int_0^x 3(x-\xi)^2 d\xi = \left[\dfrac{3(x-\xi)^3}{-3}\right]_0^x = 0-(-x^3)=x^3=$ L.H.S.

$\therefore$ $u(x)=3$ is a solution of the given integral equation.

Example 2. $u(x)+\int_0^1 x(e^{x\xi}-1)u(\xi)d\xi=e^x-x$; $u(x)=1$

Solution : Fredholm non-homogeneous integral equation of second kind

L.H.S. $=1+\int_0^1 x(e^{x\xi}-1)\cdot 1 d\xi$

$=1+x\int_0^1 (e^{\xi}-1)d\xi \qquad =1+x\left[\dfrac{e^{x\xi}}{x}-\xi\right]_0^1 = 1+x\left[\dfrac{e^x}{x}-1-\dfrac{1}{x}\right]$

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$= 1 + e^x - x - 1 = e^x - x =$ R.H.S.

$\therefore \quad u(x) = 1$ is a solution of the given integral equation.

1. VERIFICATION

EXERCISE 1.1

Write down the type of each of the following integral equations and also verify that the given functions are solution of the corresponding integral equations.

1. $u(x) = 1 - x \; ; \int_0^x e^{x-\xi} \, u(\xi) d\xi = x$

2. $u(x) = x - \frac{x^3}{6} \; ; u(x) = x - \int_0^x \sin h(x-\xi) \, u(\xi) d\xi$

3. $u(x) = xe^x \; ; u(x) = \sin x + 2\int_0^x \cos(x-\xi) \, u(\xi) d\xi$

4. $u(x) = \left(1+x^2\right)^{-3/2} \; ; u(x) = \frac{1}{1+x^2} - \int_0^x \frac{\xi}{1+x^2} \, u(\xi) d\xi$

5. $u(x) = e^x \left(2x - \frac{2}{3}\right) \; ; u(x) + 2\int_0^1 e^{x-\xi} u(\xi) d\xi = 2xe^x$

6. $u(x) = \cos 2x \; ; u(x) = \cos x + 3\int_0^{\pi} K(x,\xi) \, u(\xi) d\xi$ where $K(x,\xi) = \begin{cases} \sin x \cos \xi & , 0 \le x \le \xi \\ \cos x \sin \xi & , \xi \le x \le \pi \end{cases}$

2. CONVERSION

Initial Value Problem (I.V.P) : Differential equation + conditions at one fixed point.

Boundary Value Problem (B.V.P) : Differntial equation + conditions at more than one point.

Formula : 1. $\int_a^x \int_a^{x_2} f(x_1) dx_1 \, dx_2 = \frac{1}{\lfloor 1} \int_a^x (x-\xi) \, f(\xi) d\xi$

2. $\int_a^x \int_a^{x_3} \int_a^{x_2} f(x_1) dx_1 \, dx_2 \, dx_3 = \frac{1}{\lfloor 2} \int_a^x (x-\xi)^2 \, f(\xi) d\xi$

3. $\int_a^x \int_a^{x_n} \ldots\ldots \int_a^{x_3}\int_a^{x_2} f(x_1)\,dx_1\,dx_2\ldots..dx_{n-1}\,dx_n \quad = \frac{1}{\lfloor n-1}\int_a^x (x-\xi)^{n-1} f(\xi)\,d\xi$

Example 1. Convert the following I.V.P. into V.I.E. : $y' - y = 0$; $y(0) = 1$

Solution : $y(x) - y(0) - \int_0^x y(\xi)\,d\xi = 0$

$\Rightarrow \qquad y(x) - 1 - \int_0^x y(\xi)\,d\xi = 0$

$\Rightarrow \qquad y(x) = 1 + \int_0^x y(\xi)\,d\xi = 0$

OR

$y'(x) = u(x)$

$\Rightarrow \qquad y(x) - y(0) = \int_0^x u(\xi)\,d\xi = 0$

$\Rightarrow \qquad y(x) = 1 + \int_0^x u(\xi)\,d\xi = 0$

$\qquad u(x) - 1 - \int_0^x u(\xi)\,d\xi = 0$

$\Rightarrow \qquad u(x) = 1 + \int_0^x u(\xi)\,d\xi = 0$

Example 2. $y''(x) - 3y'(x) + 2y(x) = 4\sin x$; $y(0) = 1,\ y'(0) = -2$

Solution : $y'(x) - y'(0) - 3y(x) + 3y(0) + 2\int_0^x y(\xi)\,d\xi = \int_0^x 4\sin\xi\,d\xi$

$\Rightarrow \qquad y'(x) + 2 - 3y(x) + 3 + 2\int_0^x y(\xi)\,d\xi = -4(\cos x - 1)$

$\Rightarrow \qquad y'(x) - 3y(x) + 3 + 2\int_0^x y(\xi)\,d\xi = -4\cos x - 1$

$\Rightarrow \qquad y(x) - 1 - 3\int_0^x y(\xi)\,d\xi + 2\int_0^x (x-\xi)\,y(\xi)\,d\xi = -4\sin x - x$

$\Rightarrow \qquad y(x) = 1 - x - 4\sin x + \int_0^x (3 - 2x + 2\xi)\,y(\xi)\,d\xi$

OR

$y''(x) = u(x)$

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$$\Rightarrow \quad y'(x)+2=\int_0^x u(t)\,dt$$

$$\Rightarrow \quad y(x)-1+2x=\int_0^x (x-t)u(t)\,dt$$

$$\Rightarrow \quad y(x)=1-2x+\int_0^x (x-t)u(t)\,dt$$

$$\therefore \quad u(x)-3\int_0^x u(t)\,dt+6+2-4x+2\int_0^x (x-t)u(t)\,dt=4\sin x$$

$$\Rightarrow \quad u(x)=-8+4x+4\sin x+\int_0^x (2t-2x+3)u(t)\,dt$$

$$\Rightarrow \quad u(x)=4(x+\sin x-2)+\int_0^x \left[3-2(x-t)\right]u(t)\,dt$$

EXERCISE 2.1

Convert the following initial value problems into the corresponding Volterra integral equations :

1. $y''+y=\cos x\ ;\ y(0)=0,\ y'(0)=1$
2. $y''-5y'+6y=0\ ;\ y(0)=0,\ y'(0)=-1$
3. $y''+y=0\ ;\ y(0)=0,\ y'(0)=1.$
4. $y''+y=\cos x\ ;\ y(0)=0,\ y'(0)=0$
5. $y''+xy'+y=0\ : y(0)=1,\ y'(0)=0$
6. $\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}-3y=0\ : y(0)=1,\ y'(0)=0$
7. $\dfrac{d^2y}{dx^2}+xy=1\ ;\ y(0)=y'(0)=0.$
8. $\dfrac{d^2y}{dx^2}-\sin x\left(\dfrac{dy}{dx}\right)+e^x y=x;\ y(0)=1, y'(0)=-1$
9. $y''+\left(1+x^2\right)y=\cos x\ ;\ y(0)=0, y'(0)=2$
10. $y'''-2xy=0\ ;\ y(0)=\dfrac{1}{2},\ y'(0)=y''(0)=1$
11. $y'''+xy''+\left(x^2-x\right)y=x\,e^x+1\ ;\ y(0)=1=y'(0),\ y''(0)=0\ .$
12. $\dfrac{d^2y}{dx^2}+a_1(x)\dfrac{dy}{dx}+a_2(x)y=F(x)\ ;\ y(0)=C_0$ and $y'(0)=C_1$.
13. $y''(x)+\lambda y(x)=F(x)\ ;\ y(0)=0, y'(0)=0$

Answers

1. $y(x)=1-\cos x+x-\int_0^x (x-\xi)\ y(\xi)d\xi$ or $u(x)=\cos x-x-\int_0^x (x-t)\ u(t)dt$

2. $y(x)=-x-\int_0^x (6x-6\xi-5)\ y(\xi)d\xi$ or $u(x)=6x-5+\int_0^x (5-6x+6t)\ u(t)dt$

3. $y(x)=x-\int_0^x (x-\xi)\ y(\xi)d\xi$ or $u(x)=-x-\int_0^x (x-t)\, u(t)dt$

4. $y(x)=1-\cos x-\int_0^x (x-\xi)\ y(\xi)d\xi$ or $u(x)=\cos x-\int_0^x (x-t)\ u(t)dt$

5. $y(x)=1-\int_0^x \xi\ y(\xi)d\xi$ or $u(x)=-1-\int_0^x (2x-t)\, u(t)dt$

6. $y(x)=1+\int_0^x (x+\xi)\ y(\xi)d\xi$ or $u(x)=3+\int_0^x (5x-3t)\, u(t)dt$

7. $y(x)=\dfrac{x^2}{2}-\int_0^x (x-\xi)\xi\ y(\xi)d\xi$ or $u(x)=1-\int_0^x x(x-t)\, u(t)dt$

8. $y(x)=\left(1-x+\dfrac{x^3}{6}\right)-\int_0^x \left[(x-\xi)e^{\xi}+(x-\xi)\cos\xi-\sin\xi\right] y(\xi)d\xi$

   or $u(x)=x-\sin x-e^x(1-x)+\int_0^x \left[\sin x-e^x(x-t)\right]u(t)dt$

9. $y(x)=1-\cos x+2x-\int_0^x (x-\xi)\left(1+\xi^2\right) y(\xi)d\xi$ or $u(x)=\cos x-2x\left(1+x^2\right)-\int_0^x \left(1+x^2\right)(x-t)\ u(t)dt$

10. $y(x)=\dfrac{1}{2}+x+\dfrac{x^2}{2}+\int_0^x \xi(x-\xi)^2\ y(\xi)d\xi$ or $u(x)=x(x+1)^2+\int_0^x x(x-t)^2 u(t)dt$

11. $y(x)=4+(x-3)e^x+3x+\dfrac{x^3}{6}+\int_0^x \left[(-3\xi+2x)+\dfrac{1}{2}(x-\xi)^2\left(\xi-\xi^2\right)\right] y(\xi)d\xi$

    or $u(x)=xe^x+1-x\left(x^2-1\right)-\int_0^x \left[x+\dfrac{1}{2}\left(x^2-x\right)(x-t)^2\right]u(t)dt$

12. $y(x)=c_0+c_1x+a_1(0)\,c_0x+\int_0^x (x-\xi)\ F(\xi)d\xi-\int_0^x \left[a_1(\xi)+(x-\xi)\left(a_2(\xi)-a_1{}'(\xi)\right)\right]y(\xi)d\xi$

    or $u(x)=F(x)-C_1a_1(x)-(C_0+C_1x)a_2(x)-\int_0^x \left[a_1(x)+a_2(x)(x-t)\right]u(t)dt$

13. $y(x)=\int_0^x (x-\xi)\ F(\xi)d\xi-\int_0^x \lambda(x-\xi)\ y(\xi)d\xi$ or $u(x)=F(x)-\lambda\int_0^x (x-t)\ u(t)dt$

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Leibnitz Rule : $\frac{d}{dx}\left(\int_{a(x)}^{b(x)} F(x,\alpha)\,d\alpha\right)=\int_{a(x)}^{b(x)}\frac{d}{dx}\big(F(x,\alpha)\big)d\alpha+F\big(x,b(x)\big)\frac{d}{dx}\big(b(x)\big)-F\big(x,a(x)\big)\frac{d}{dx}\big(a(x)\big)$

Derive the original differential equation with the initial conditions from the integral equation :

Example 1. $y(x)=\frac{x^3}{6}-x+1+\int_0^x\left[\sin t-(x-t)\left(e^t+\cos t\right)\right]y(t)\,dt$

Solution : $y'(x)=\frac{x^2}{2}-1+x+\int_0^x -\left(e^t+\cos t\right)y(t)\,dt+\sin x y(x)$

$\Rightarrow \quad y''(x)=x+1-\int_0^x 0\cdot y(t)\,dt-\left(e^x+\cos x\right)y(x)+\sin x\, y'(x)+\cos x\, y(x)$

$\Rightarrow \quad y''(x)-\sin x\cdot y'(x)+e^x y(x)=x\ ;\ y(0)=1,\ y'(0)=-1$

EXERCISE 2.2

Derive the original differential equation with the initial conditions from the integral equation:

1. $y(x)=1-x-4\sin x+\int_0^x\left[3-2(x-t)\right]\ y(t)\,dt$

2. $y(x)=1+\int_0^x (x+t)\ y(t)\,dt$

3. (a) Show that if $y(x)$ satisfies the differential equation $y''(x)+xy=1$ and the conditions

   $y(0)=y'(0)=0$, then y also satisfy the Volterra equation $y(x)=\frac{1}{2}x^2+\int_0^x t(t-x)\ y(t)\,dt$

   (b) Prove that the converse of the preceding statement is also true.

4. Show that the solution of the Volterra equation $y(x)=1+\int_0^x(\xi-x)\ y(\xi)\,d\xi$ satisfies the differential equation $y''(x)+y(x)=0$ and the boundary conditions $y(0)=1$, $y'(0)=0$.

Answers

1. $y''(x)-3y'(x)+2y(x)=4\sin x\ ;\ y(0)=1,\ y'(0)=-2$

3. $y''(x)-2xy'(x)-3y(x)=0\ ;\ y(0)=1,\ y'(0)=0$

Reduce the boundary value problem into an Fredholm integral equation :

Example 1. $y''+y=0$; $y(0)=1,\ y'(1)=0$

Solution : $y'(x)-y'(0)+\int_0^x y(\xi)\,d\xi=0$

$\Rightarrow \quad y'(x)-c+\int_0^x y(\xi)\,d\xi=0$

$\Rightarrow \quad y'(1)-c+\int_0^x y(\xi)\,d\xi=0$

$\Rightarrow \quad c=\int_0^1 y(\xi)\,d\xi$

$\Rightarrow \quad y(x)-y(0)-cx+\int_0^x (x-\xi)\,y(\xi)\,d\xi=0$

$\Rightarrow \quad y(x)-1-\int_0^x x\,y(\xi)\,d\xi-\int_x^1 x\,y(\xi)\,d\xi+\int_0^x (x-\xi)\,y(\xi)\,d\xi=0$

$\Rightarrow \quad y(x)-1-\int_0^x \xi\,y(\xi)\,d\xi-\int_x^1 x\,y(\xi)\,d\xi=0$

$\Rightarrow \quad y(x)=1+\int_0^x \xi\,y(\xi)\,d\xi+\int_x^1 x\,y(\xi)\,d\xi$

EXERCISE 2.3

Reduce the following boundary value problem into an Fredholm integral equation :

1. $y''(x)+\lambda\,y(x)=0$; $y(0)=0$, $y(l)=0$

2. $\dfrac{d^2y}{dx^2}+xy=1$; $y(0)=y(1)=1$

3. $u''(x)+\lambda u=0$; $u(0)=0,\ u(1)=0$

4. $y''(x)+y=x$; $y(0)=0$, $y'(1)=0$

5. $y''(x)+\lambda y=x$; $y(0)=y(\pi)=0$

6. $y''+\lambda y=0$; $y(0)=y\left(\dfrac{\pi}{2}\right)=0$

Answers

1. $y(x)=\lambda\int_0^l K(x,t)\;y(t)\,dt$, where $K(x,t)=\begin{cases}\dfrac{t(l-x)}{l}, & if\ 0<t<x\\[2mm] \dfrac{x(l-t)}{l}, & if\ x<t<l\end{cases}$

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2. $y(x)=1+\frac{1}{2}x(x-1)+\int_0^x t^2(1-x)\ y(t)dt+\int_x^1 xt(1-t)y(t)dt$

3. $u(x)=\lambda\int_0^1 K(x,t)u(t)dt$, where $K(x,t)=\begin{cases}(1-t)x, & 0\le x\le t\\ (1-x)t, & t\le x\le 1\end{cases}$

4. $y(x)=\frac{1}{6}(x^3-3x)+\int_0^1 K(x,t)\,y(t)dt$, where $K(x,t)=\begin{cases}x, & x<t\\ t, & x>t\end{cases}$

5. $y(x)=\frac{1}{6}x\left(x^2-\pi^2\right)+\lambda\int_0^{\pi} K(x,t)\,y(t)dt$, where $K(x,t)=\begin{cases}\frac{x}{\pi}(x-t) & \text{when } t>x\\ \frac{t}{\pi}(\pi-x) & \text{when } t<x\end{cases}$

6. $y(x)=\lambda\int_0^{\frac{\pi}{2}} K(x,t)\ y(t)dt$, where $K(x,t)=\begin{cases}\left[1-(2t/\pi)\right]x, & 0\le x<t\\ \left[1-(2x/\pi)\right]t, & t<x\le\pi/2\end{cases}$

EXERCISE 2.4

Convert Fredholm Integral equation into Boundary Value Problem :

1. $y(x)=\int_0^l \frac{\lambda x(l-t)}{l}\ y(t)dt-\int_0^x \lambda(x-t)\,y(t)dt$

2. $y(x)=\int_0^x (x-t)\ F(t)dt-x\int_0^1 (1-t)\ F(t)dt$

Answers

1. $y''(x)+\lambda y(x)=0$; $y(0)=0$, $y(l)=0$

2. $y''(x)=F(x)$; $y(0)=0$, $y(1)=0$

3. Evaluation of Resolvent Kernel For Volterra Integral equations

Formula for calculating Itarated kernels Resolvent kernel

(i) $K_1(x,\xi)=K(x,\xi)$

$$K_{n+1}(x,\xi)=\int_\xi^x K(x,t)\ K_n(t,\xi)\,dt$$

These K_n's are called Iterated Kernels.

(ii) $R(x,\xi;\lambda)=K_1+\lambda K_2+\lambda^2K_3+.....+\lambda^{n-1}K_n+.......=\sum_{m=1}^{\infty}\lambda^{m-1}K_m$

This series is known as **Neumann's Series** and $R(x,\xi;\lambda)$ is called **Resolvent Kernel**.

Remark : $\int_{\xi}^{x} K(x,t)K_n(t,\xi)\,dt = \int_{\xi}^{x} K_n(x,t)K(t,\xi)\,dt$.

Find the resolvent kernel $R(x,\xi;\lambda)$ of the following kernels $K(x,\xi)$:

Example 1. $K(x,\xi)=e^{x-\xi}$

Solution : $K(x,t)\cdot K(t,\xi)=e^{x-t}\cdot e^{t-\xi}=e^{x-\xi}=K(x,\xi)$

$\therefore \quad R(x,\xi;\lambda)=e^{x-\xi}\cdot e^{\lambda(x-\xi)}$

Example 2. $K(x,\xi)=a^{x-\xi}$

Solution : $K(x,t)\cdot K(t,\xi)=a^{x-t}\cdot a^{t-\xi}=a^{x-\xi}=K(x,\xi)$

$\therefore \quad R(x,\xi;\lambda)=a^{x-\xi}\cdot e^{\lambda(x-\xi)}$

EXERCISE 3.1

Find the resolvent kernel $R(x,\xi;\lambda)$ of the following kernels $K(x,\xi)$.

1. 1
2. $3^{x-\xi}$
3. $e^{x^2-\xi^2}$
4. $\dfrac{\cosh x}{\cosh \xi}$
5. $\dfrac{2+\cos x}{2+\cos\xi}$
6. $\dfrac{1+x^2}{1+\xi^2}$
7. $e^{\xi-x}$

Answers

1. $e^{\lambda(x-\xi)}$
2. $3^{x-\xi}.e^{\lambda(x-\xi)}$
3. $e^{x^2-\xi^2}\,.\,e^{\lambda(x-\xi)}$
4. $\dfrac{\cos h\,x}{\cos h\,\xi}.e^{\lambda(x-\xi)}$
5. $\dfrac{2+\cos x}{2+\cos\xi}.e^{\lambda(x-\xi)}$
6. $\dfrac{1+x^2}{1+\xi^2}.e^{\lambda(x-\xi)}$
7. $e^{\xi-x}.e^{\lambda(x-\xi)}$

Formula for calculating Resolvent kernel when $K(x,\xi)$ is a polynomial in $\xi\,(or\,x)$.

Case I : $K(x,\xi)=a_0(x)$

Consider the differential equation $\quad \dfrac{dy}{dx}-\lambda\, a_0(x)\,y=0 \qquad$ .....(1)

and the auxiliary condition $\quad y(\xi)=1 \qquad$ .....(2)

Let $y(x)$ be the solution of (1) and (2) , then the Resolvent Kernel is given by $R(x,\xi;\lambda)=\dfrac{1}{\lambda}\dfrac{dy}{dx}$

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Case II : $K(x,\xi) = a_0(x) + a_1(x)(x-\xi)$

Consider the differential equation $\dfrac{d^2y}{dx^2} - \lambda\left[a_0(x)\dfrac{dy}{dx} + a_1(x)y\right] = 0$ ......(1)

and the auxiliary conditions $y(\xi) = 0\ ,\ y'(\xi) = 1$ ......(2)

Let $y(x)$ be the solution of (1) and (2) , then the Resolvent Kernel is given by $R(x,\xi;\lambda) = \dfrac{1}{\lambda}\dfrac{d^2y}{dx^2}$

Case III : $K(x,\xi) = a_0(x) + a_1(x)(x-\xi) + a_2(x)\dfrac{(x-\xi)^2}{2!}$

Consider the differential equation $\dfrac{d^3y}{dx^3} - \lambda\left[a_0(x)\dfrac{d^2y}{dx^2} + a_1(x)\dfrac{dy}{dx} + a_2(x)y\right] = 0$ .....(1)

and the auxiliary conditions $y(\xi) = y'(\xi) = 0\ ,\ y''(\xi) = 1$ .....(2)

Let $y(x)$ be the solution of (1) and (2) , then the Resolvent Kernel is given by $R(x,\xi;\lambda) = \dfrac{1}{\lambda}\dfrac{d^3y}{dx^3}$

Note : If we have $\xi - x$,then consider it as $-(x-\xi)$ and solve using above formula.

Find out Resolvent kernel for the following kernels and given value of parameter

Example 1. $K(x,\xi) = 2x\ ;\ \lambda = 1$

Solution : $a_0(x) = 2x$

$\dfrac{dy}{dx} - \lambda a_0(x)y = 0\ ,\ y(\xi) = 1$

$\Rightarrow \quad \dfrac{dy}{dx} - 2xy = 0$

$\Rightarrow \quad \dfrac{dy}{dx} - 2xy \qquad \Rightarrow \quad \displaystyle\int \frac{dy}{y} = \int 2x\,dx$

$\Rightarrow \quad \log y = x^2 + \log c \qquad \Rightarrow \quad y = c\cdot e^{x^2}$

$y(\xi) = 1 \qquad \Rightarrow \quad c\cdot e^{\xi^2} = 1 \qquad \Rightarrow \quad c = e^{-\xi^2}$

$\therefore \quad y(x) = e^{x^2-\xi^2}$

$R(x,\xi;1) = \dfrac{\lambda}{1}\dfrac{dy}{dx} = 2xe^{x^2-\xi^2}$

Example 2. $K(x,\xi)=x-\xi$; $\lambda=1$

Solution : $a_0(x)=0,\ a_1(x)=1$

$$\therefore \quad \frac{d^2y}{dx^2}-\lambda\left[a_0(x)\frac{dy}{dx}+a_1(x)y\right]=0$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}-y=0 \ ; \quad y(\xi)=0,\ y'(\xi)=1$$

$$\Rightarrow \quad D^2-1=0 \quad \Rightarrow \quad D=\pm 1$$

$$\Rightarrow \quad y(x)=c_1e^{x}-c_2e^{-x}$$

$y(\xi)=0$ and $y'(\xi)=1$

$$\Rightarrow \quad c_1e^{\xi}+c_2e^{-\xi}=0 \qquad c_1e^{\xi}-c_2e^{-\xi}=1$$

$$\Rightarrow \quad 2c_1e^{\xi}=1 \quad \Rightarrow \quad c_1=\frac{1}{2}e^{-\xi} \quad \text{and} \quad c_2=\frac{-1}{2}e^{\xi}$$

$$\therefore \quad y(x)=\frac{1}{2}\left(e^{x-\xi}-e^{\xi-x}\right) \quad \Rightarrow \quad y(x)=\sinh(x-\xi)$$

$$R(x,\xi;\lambda)=\frac{1}{\lambda}\frac{d^2y}{dx^2}=\sinh(x-\xi)$$

EXERCISE 3.2

Find out Resolvent Kernel for the following kernels and given value of parameter.

1. $K(x,\xi)=x-\xi$; $\lambda=\lambda$
2. $K(x,\xi)=\xi-x$; $\lambda=1$
3. $K(x,\xi)=2-(x-\xi)$; $\lambda=1$
4. $K(x,\xi)=-2+3(x-\xi)$; $\lambda=1$
5. $K(x,\xi)=5-6(x-\xi)$; $\lambda=1$
6. $K(x,\xi)=3+6(x-\xi)-4(x-\xi)^2$; $\lambda=1$

Answers

1. $\frac{1}{\sqrt{\lambda}}\sin h\sqrt{\lambda}(x-\xi)$
2. $\sin(\xi-x)$
3. $(x-\xi+2)\,e^{x-\xi}$
4. $\frac{1}{4}e^{x-\xi}-\frac{9}{4}\,e^{-3(x-\xi)}$
5. $9\,e^{3(x-\xi)}-4e^{2(x-\xi)}$
6. $-\frac{1}{9}\,e^{x-\xi}-\frac{4}{9}\,e^{-2(x-\xi)}+\frac{32}{9}\,e^{4(x-\xi)}$

Example 1. $K(x,\xi)=\xi$; $\lambda=\frac{1}{2}$

Solution : $K_1(x,\xi)=\xi$; $\lambda=\frac{1}{2}$

$$K_2(x,\xi)=\int_{\xi}^{x}K(x,t),\,K_1(t,\xi)\,dt$$

$$= \int_{\xi}^{x} t\cdot\xi \, dt \qquad = \xi\left(\frac{t^2}{2}\right)_{\xi}^{x} \qquad = \frac{\xi}{2}\left(x^2-\xi^2\right)$$

$$K_3(x,\xi) = \int_{\xi}^{x} t\cdot\frac{\xi}{2}\left(t^2-\xi^2\right)dt$$

$$= \frac{\xi}{2}\int_{\xi}^{x}\left(t^3-\xi^2 t\right)dt$$

$$= \frac{\xi}{2}\left[\frac{t^4}{4}-\frac{\xi^2 t^2}{2}\right]_{\xi}^{x}$$

$$= \frac{\xi}{2}\left[\frac{t^4-2t^2\xi^2}{4}\right]_{\xi}^{x}$$

$$= \frac{\xi}{8}\left[x^4-\xi^4-2\xi^2\left(x^2-\xi^2\right)\right]$$

$$= \frac{\xi}{8}\left[x^4+\xi^4-2x^2\xi^2\right] = \frac{\xi}{\lfloor 2}\left(\frac{x^2-\xi^2}{2}\right)^2$$

Parallely

$$K_4(x,\xi) = \frac{\xi}{\lfloor 3}\left(\frac{x^2-\xi^2}{2}\right)^3$$

$$R(x,\xi;\lambda) = K_1+\lambda K_2+\lambda^2 K_3+\ldots.$$

$$\xi+\frac{1}{2}\xi\left(\frac{x^2-\xi^2}{2}\right)+\frac{1}{2^2}\frac{\xi}{2}\left(\frac{x^2-\xi^2}{2}\right)^2+\ldots..$$

$$\xi\left[1+\left(\frac{x^2-\xi^2}{4}\right)+\frac{1}{\lfloor 2}\left(\frac{x^2-\xi^2}{4}\right)^2+\ldots.\right] \qquad = \xi\cdot e^{\frac{x^2-\xi^2}{4}}$$

EXERCISE 3.3

Miscellaneous type :

1. $K(x,\xi) = x\xi \;,\; \lambda = 1$

Answers

1. $x\xi.e^{\frac{x^3-\xi^3}{3}}$

4. Solution of Volterra Integral Equation with the help of Resolvent Kernel

Formula : Let the given Volterra equation is $u(x)= f(x)+\lambda\int_0^x K(x,\xi)u(\xi)d\xi$ ...... (1)

Suppose that the corresponding Resolvent Kernel is $R(x,\xi;\lambda)$ then the solution of (1) is given by

$$u(x)= f(x)+\lambda\int_0^x R(x,\xi;\lambda)\, f(\xi)d\xi .$$

Solve the following integral equations :

Example 1. $\phi(x)=(1+x)-\int_0^x\phi(\xi)d\xi$

Solution : $K(x,\xi)=1,\ \ \lambda=-1$

$$R(x,\xi;\,\lambda)=e^{\lambda(x-\xi)}=e^{-(x-\xi)}=e^{\xi-x}$$

$$\phi(x)=(1+x)-\int_0^x e^{\xi-x},(1+\xi)d\xi$$

$$=(1+x)-e^{-x}\int_0^x(1+\xi)\,e^{\xi}\,d\xi$$

$$=(1+x)-e^{-x}\left[(1+\xi)e^{\xi}-e^{\xi}\right]_0^x$$

$$=(1+x)-(1+x)+1$$

$$(1+x)-e^{-x}\left[(1+x)e^{x}-e^{x}-1+\lambda\right]$$

$$=(1+x)-(1+x)+1\ \ =1$$

$\therefore \qquad \phi(x)=1$

Example 2. $\phi(x)=e^x+\int_0^x e^{x-\xi}\phi(\xi)d\xi$

Solution : $K(x,\xi)=e^{x-\xi},\quad \lambda=1$

$$R(x,\xi;\,\lambda)=e^{x-\xi}\cdot e^{x-\xi}=e^{2(x-\xi)}$$

$$\phi(x)=e^x+\int_0^x e^{2(x-\xi)}\cdot e^{\xi}\,d\xi$$

$$=e^x+e^{2x}\int_0^x e^{-\xi}\,d\xi \qquad \Rightarrow \qquad e^x+e^{2x\left(-e^{-\xi}\right)_0^x}$$

$$\Rightarrow e^x-e^{2x}\left(e^{-x}-1\right)=e^{2x}$$

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EXERCISE 4.1

Solve the following integral equations:

1. $\phi(x)=1+\int_0^x \phi(\xi)d\xi$.

2. (i) $\phi(x)=\sin x+2\int_0^x e^{x-\xi}\phi(\xi)d\xi$ (ii) $\phi(x)=e^x-\int_0^x e^{x-\xi}\phi(\xi)d\xi$

3. $\phi(x)=1+\int_0^x a^{x-\xi}\phi(\xi)d\xi$ 4. $\phi(x)=x.3^x-\int_0^x 3^{x-\xi}\phi(\xi)d\xi$

5. (i) $\phi(x)=e^{x^2}+\int_0^x e^{x^2-\xi^2}\phi(\xi)d\xi$ (ii) $\phi(x)=e^{x^2+2x}+2\int_0^x e^{x^2-\xi^2}\phi(\xi)d\xi$ (iii) $\phi(x)=1-2x-\int_0^x e^{x^2-\xi^2}\phi(\xi)d\xi$

6. $\phi(x)=e^x\sin x+\int_0^x \frac{2+\cos x}{2+\cos\xi}\phi(\xi)d\xi$ 7. $\phi(x)=1+x^2+\int_0^x \frac{1+x^2}{1+\xi^2}\phi(\xi)d\xi$

8. $\phi(x)=x.e^{\frac{x^2}{2}}+\int_0^x e^{-(x-\xi)}\phi(\xi)d\xi$

Answers

1. $\phi(x)=e^x$

2. (i) $\phi(x)=\frac{1}{5}e^{3x}+\frac{2}{5}\sin x-\frac{1}{5}\cos x$ (ii) $\phi(x)=1$

3. $\frac{\log a+a^x e^x}{1+\log a}$ 4. $\phi(x)=3^x\left(1-e^{-x}\right)$

5. (i) $\phi(x)=e^{x^2+x}$ (ii) $\phi(x)=e^{x^2+2x}.(1+2x)$ (iii) $\phi(x)=e^{x^2-x}-2x$

6. $\phi(x)=e^x\sin x+e^x(2+\cos x)\log\frac{3}{2+\cos x}$ 7. $\phi(x)=e^x\left(1+x^2\right)$ 8. $\phi(x)=e^{\frac{x^2}{2}}(x+1)-1$

Solve the Volterra integral equation :

Example 1. $\phi(x)=x+\int_0^x (\xi-x)\phi(\xi)d\xi$

Solution : $K(x,\xi)=\xi-x,\quad \lambda=1$

$R(x,\xi;\lambda)=\sin(\xi-x)$

$$\phi(x) = x + \int_0^x \sin(\xi - x)\cdot \xi \, d\xi$$

$$= x + \left[-\xi \cos(\xi - x) + \sin(\xi - x)\right]_0^x$$

$$= x + [-x + \sin x]$$

$$= \sin x$$

$$\therefore \qquad \phi(x) = \sin x$$

EXERCISE 4.2

Solve the following Volterra integral equations :

1. $\phi(x) = 2x^2 + 2 - \frac{1}{2}\int_0^x 2x\,\phi(\xi)\,d\xi$

2. (i) $\phi(x) = (1+x) + \int_0^x (x-\xi)\,\phi(\xi)\,d\xi$ (ii) $\phi(x) = x - \int_0^x (x-\xi)\,\phi(\xi)\,d\xi$

   (iii) $\phi(x) = 1 - \int_0^x (x-\xi)\,\phi(\xi)\,d\xi$ (iv) $\phi(x) = 1 + \int_0^x (x-\xi)\,\phi(\xi)\,d\xi$

   (v) $\phi(x) = (1+x) + \lambda\int_0^x (x-\xi)\,\phi(\xi)\,d\xi$

3. (i) $\phi(x) = \cos x - x - 2 + \int_0^x (\xi - x)\,\phi(\xi)\,d\xi$ (ii) $\phi(x) = 1 + \int_0^x (\xi - x)\,\phi(\xi)\,d\xi$

4. $\phi(x) = 29 + 6x + \int_0^x \left[5 - 6(x-\xi)\right]\phi(\xi)\,d\xi$ 5. $\phi(x) = 1 - 2x - 4x^2 + \int_0^x \left[3 + 6(x-\xi) - 4(x-\xi)^2\right]\phi(\xi)\,d\xi$

Answers

1. $\phi(x) = 2$

2. (i) $\phi(x) = e^x$ (ii) $\phi(x) = \sin x$ (iii) $\phi(x) = \cos x$

   (iv) $\phi(x) = \cos hx$ (v) $\phi(x) = 1 + x + \lambda\left(\frac{x^2}{2!} + \frac{x^3}{3!}\right) + \lambda^2\left(\frac{x^4}{4!} + \frac{x^5}{5!}\right) + \lambda^3\left(\frac{x^6}{6!} + \frac{x^7}{7!}\right) + \ldots$

3. (i) $\phi(x) = -\cos x - \sin x - \frac{1}{2}x\sin x$ (ii) $\cos x$

4. $\phi(x) = 27e^{3x} - 64e^{2x}$ 5. $\phi(x) = e^x$

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EXERCISE 4.3

1. $\phi(x)=1+\int_0^x x\xi\,\phi(\xi)\,d\xi$

Answers

1. $\phi(x)=1+\frac{x^3}{2}+\frac{x^6}{2.5}+\frac{x^9}{2.5.8}+\frac{x^{12}}{2.5.8.11}+.....$

5. Solution of Volterra integral equation using Laplace transform :

Def. Laplace Transform : Let $f(t)$ be a function of *t* defined for all positive value of *t*, then the Laplace transform of $f(t)$, denoted by $L[f(t)]$ is defined as $L[f(t)]=\int_0^\infty e^{-st}f(t)\,dt$ provided that the integral exists. It is clear that $L[f(t)]$ is a function of *s* and is denoted by $F(s)$. Thus we have

$$L[f(t)]=F(s)=\int_0^\infty e^{-st}f(t)\,dt$$

Properties of the Laplace Transform :

1. If $L[f(t)]=F(s)$ then $L\left[e^{at}f(t)\right]=F(s-a)$
2. If $L[f(t)]=F(s)$, then $L[f(at)]=\frac{1}{a}F\left(\frac{s}{a}\right)$, where $a(\neq 0)$ is any constant.
3. If $L[f(t)]=F(s)$, then $L\left[t^n f(t)\right]=(-1)^n\frac{d^n}{ds^n}[F(s)]$ where *n* is a positive integer.
4. If $L[f(t)]=F(s)$, then $L\left[\frac{f(t)}{t}\right]=\int_0^\infty F(s)\,ds$
5. $L[f'(t)]=sF(s)-f(0)$
6. $L\left[f^n(t)\right]=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-.....-f^{(n-1)}(0)$
7. If $L[f(t)]=F(s)$, then $L\left[\int_0^t f(u)\,du\right]=\frac{1}{s}F(s)$
8. The function $H(t-a)=\begin{cases}0 & , \quad t<a\\ 1 & , \quad t>a\end{cases}$ is known as unit step function and its Laplace transform is

given by $L\left[H(t-a)\right]=\frac{e^{-as}}{s}$

9. If $L\left[f(t)\right]=F(s)$, then for any constant $a\geq 0$, the function

$$g(t)=f(t-a)H(t-a)=\begin{cases}0 & , \quad t<a\\ f(t-a) & , \quad t>a\end{cases} \quad \text{then } L\left[g(t)\right]=L\left[f(t-a)H(t-a)\right]=e^{-as}F(s).$$

10. $\lim\limits_{t\to 0} f(t)=\lim\limits_{s\to\infty} sF(s)$

11. $\lim\limits_{t\to\infty} f(t)=\lim\limits_{s\to\infty} sF(s)$

12. The convolution of two functions $f(t)$ and $g(t)$ is defined as

$$f(t)*g(t)=\int_0^t f(u)g(t-u)\,du=\int_0^t f(t-u)g(u)\,du$$

13. If $L^{-1}\left[F(s)\right]=f(t)$ and $L^{-1}\left[G(s)\right]=g(t)$ then

$L^{-1}\left[F(s)\cdot G(s)\right]=f(t)*g(t)=\int_0^t f(u)\cdot g(t-u)\,du$ known as convolution theorem.

Table of Laplace Transforms

$f(t)=L^{-1}\{F(s)\}$	$F(s)=L\{f(t)\}$
1. 1	$\frac{1}{s}$
2. e^{at}	$\frac{1}{s-a}$
3. $t^n,\ n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$
4. $t^p,\ p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5. $\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{1}{2}}}$
6. $t^{n-\frac{1}{2}},\ n=1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7. $\sin(at)$	$\frac{a}{s^2+a^2}$
8. $\cos(at)$	$\frac{s}{s^2+a^2}$
9. $t\sin(at)$	$\frac{2as}{\left(s^2+a^2\right)^2}$

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10. $t\cos(at)$	$\dfrac{s^2-a^2}{\left(s^2+a^2\right)^2}$
11. $\sinh(at)$	$\dfrac{a}{s^2-a^2}$
12. $\cosh(at)$	$\dfrac{s}{s^2-a^2}$
13. $e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$
14. $e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
15. $e^{at}\sinh(bt)$	$\dfrac{b}{(s-a)^2-b^2}$
16. $e^{at}\cosh(bt)$	$\dfrac{s-a}{(s-a)^2-b^2}$
17. $t^n e^{at},\ n=1,2,3,\ldots$	$\dfrac{n!}{(s-a)^{n+1}}$
18. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$
19. $e^{ct}f(t)$	$F(s-c)$
20. $t^n f(t),\ n=1,2,3,\ldots.$	$(-1)^n F^{(n)}(s)$
21. $\dfrac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$
22. $\int_0^t f(v)\,dv$	$\dfrac{F(s)}{s}$
23. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
24. $f'(t)$	$sF(s)-f(0)$

Properties of Gamma function :

The Gamma function which is defined as $\Gamma(t)=\int_0^\infty e^{-x}x^{t-1}\,dx$. If n is a positive integer then, $\Gamma(n+1)=n!$

The Gamma function is an extension of the normal factorial function. Here are a couple of quick facts for the Gamma function

(i) $\Gamma(p+1)=p\Gamma(p)$

(ii) $p(p+1)(p+2)....(p+n-1)=\dfrac{\Gamma(p+n)}{\Gamma(p)}$

(iii) $\Gamma\left(\dfrac{1}{2}\right)=\sqrt{\pi}$

(iv) $\sqrt{n}\sqrt{1-n}=\dfrac{\pi}{\sin(n\pi)}, \quad \forall\ n\in(0,1)$

Def. Difference Kernel : A kernel $K(x,t)$ is said to be difference kernel if it is of the form $K(x-t)$, where $K(x-t)$ depends only on the difference $x-t$. e.g. $e^{x-t}, \sin(x-t), (x-t)^2$ all are difference kernels.

Example 1 : Solve the integral equation $x=\int_0^x e^{x-\xi}\phi(\xi)\,d\xi$

Solution : The integral may be expressed as $x=\phi(x)*e^x$ …..(1)

Taking the Laplace transform of both the sides of (1), we have $L(x)=L\left[\phi(x)*e^x\right]$

$\Rightarrow \quad \dfrac{1}{S^2}=L\left[\phi(x)\right]\cdot L\left[e^x\right] \qquad \Rightarrow \quad \dfrac{1}{S^2}=L\left[\phi(x)\right]\cdot\dfrac{1}{s-1}$

$\Rightarrow \quad L\left[\phi(x)\right]=\dfrac{s-1}{S^2}=\dfrac{1}{s}-\dfrac{1}{s^2}$ …..(2)

Taking inverse Laplace transform of (2), we have $\phi(x)=L^{-1}\left(\dfrac{1}{s}\right)-L^{-1}\left(\dfrac{1}{s^2}\right) \quad \Rightarrow \quad \phi(x)=1-x$

Example 2 : Solve the integral equation $\phi(x)=x^2+\int_0^x \sin(x-\xi)\phi(\xi)\,d\xi$

Solution : The integral may be expressed as $\phi(x)=x^2+\phi(x)*\sin x$ …..(1)

Taking Laplace transform we have $L\left[\phi(x)\right]=L\left(x^2\right)+L\left[\phi(x)*\sin x\right]$

$\Rightarrow \quad L\left[\phi(x)\right]=\dfrac{\lfloor 2}{s^3}+L\left[\phi(x)\right]\cdot L(\sin x) \qquad \Rightarrow \quad L\left[\phi(x)\right]=\dfrac{2}{s^3}+L\left[\phi(x)\right]\cdot\dfrac{1}{s^2+1}$

$\Rightarrow \quad L\left[\phi(x)\right]=\dfrac{2\left(s^2+1\right)}{s^5}=\dfrac{2}{s^3}+\dfrac{2}{s^5}$

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Taking inverse Laplace transform, we have $\phi(x)=L^{-1}\left(\frac{2}{s^3}\right)+L^{-1}\left(\frac{2}{s^5}\right)$

$\Rightarrow \quad \phi(x)=2\cdot\frac{x^2}{\lfloor 2}+2\cdot\frac{x^4}{\lfloor 4} \qquad \Rightarrow \quad \phi(x)=x^2+\frac{1}{12}x^4$

EXERCISE 5.1

Solve the following volterra integral equation :

1. $\int_0^x \frac{\phi(\xi)}{(x-\xi)^{\frac{1}{3}}}d\xi = x(1+x)$
2. $\int_0^x \phi(\xi)\phi(x-\xi)d\xi = 16\sin 4x$
3. $\phi(x)=1-\int_0^x (x-\xi)\phi(\xi)d\xi$
4. $\phi(x)=x+2\int_0^x \cos(x-\xi)\phi(\xi)d\xi$
5. $\phi(x)=e^{-x}-2\int_0^x \cos(x-\xi)\phi(\xi)d\xi$
6. $\phi'(x)=\sin x+\int_0^x \cos\xi\,\phi(x-\xi)d\xi, \phi(0)=0$

Answers

1. $\phi(x)=\frac{3\sqrt{3}}{4\pi}x^{\frac{1}{3}}(2+3x)$
2. $\phi(x)=\pm 8J_0(4x)$
3. $\phi(x)=\cos x$
4. $\phi(x)=2e^x(x-1)+x+2$
5. $\phi(x)=e^{-x}(1-2x+x^2)$
6. $\phi(x)=\frac{x^2}{2}$

6. Solution of Fredholm Integral Equation with Separable (degenerate) Kernel

Def. Separable (degenerate) Kernel : A Kernel $K(x,t)$ is said to be separable or degenerate if it can be expressed as $K(x,t)=\sum_{i=1}^{n} a_i(x)\, b_i(t)$ **(terms should be finite)**

$$=a_1(x)\,b_1(t)+a_2(x)\,b_2(t)+........+a_n(x)\,b_n(t)$$

Illustration: Which of the following Kernels are separable:

(i) $2xt+3x^2t^2$ (ii) $1-3xt$ (iii) $\cos(x-t)$ (iv) e^{xt}

(v) $xt+\sin(x-t)+e^{xt}$ (vi) x^2t+xt^2 (vii) $\sin(x+t)$ (viii) e^{x-t}

(ix) $\sin(xt)$ (x) $\cos(xt)+xt$

Def . Eigen Values or characteristic values : Consider a homogeneous Fredholm Integral equation of the second kind

$$\phi(x) = \lambda \int_a^b K(x,t)\,\phi(t)\,dt \qquad \text{.........(1)}$$

Then (1) always has a obvious solution $\phi(x)=0$, which is known as zero-solution or trivial solution of (1)

The values of the parameter λ for which (1) has a non-zero (non-trivial) solution are called eigen values (or characteristic values) of (1).

Def. Eigen functions or characteristic functions : Let λ_0 be an eigen value of (1) and $\phi_0(x)$ be the corresponding non-trivial solution of (1), the function $\phi_0(x)$ is called the eigen function or characteristic function of (1) corresponding to $\lambda=\lambda_0$.

Example 1. Determine the eigen values and eigen functions (if exists) of the Fredholm homogeneous integral Equation. $\phi(x)=\lambda\int_0^{\pi/4}\sin^2 x\,\phi(t)\,dt$

Solution : $\phi(x)=\lambda\sin^2 x\cdot c,\;\; c=\int_0^{\frac{\pi}{4}}\phi(t)\,dt$

$$c=\int_0^{\frac{\pi}{4}}\lambda c\sin^2 t\,dt \;=\frac{\lambda c}{2}\int_0^{\frac{\pi}{4}}(1-\cos 2t)\,dt$$

$$\Rightarrow \qquad c=\frac{\lambda c}{2}\left[t-\frac{\sin 2t}{2}\right]_0^{\frac{\pi}{4}}$$

$$c=\frac{\lambda c}{2}\left[\frac{\pi}{4}-\frac{1}{2}\right] \qquad\qquad \Rightarrow \qquad c\left[1-\frac{\lambda}{2}\left(\frac{\pi-2}{4}\right)\right]=0$$

$$\Rightarrow \qquad \left[1-\frac{\lambda}{8}(\pi-2)\right]c=0$$

$$1-\frac{\lambda}{8}(\pi-2)=0 \;\Rightarrow\; \lambda=\frac{8}{\pi-2}$$

$$\therefore \qquad \phi(x)=\frac{8c}{\pi-2}\sin^2 x=c\sin^2 x$$

Values of λ	No. of solution	solution
$\lambda=\frac{8}{\pi-2}$	infinite	$\phi(x)=c\sin^2 x$
$\lambda\neq\frac{8}{\pi-2}$	unique	$\phi(x)=0$

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EXERCISE 6.1

Solve the following Fredholm homogeneous Integral equations of second kind :

1. $\phi(x) = -\int_0^1 \phi(t)\,dt$
2. $\phi(x) = \frac{1}{2}\int_0^{\pi} \sin x\ \phi(t)\,dt$
3. $\phi(x) = \frac{1}{50}\int_0^{10} t\ \phi(t)\,dt$
4. $\phi(x) = \frac{1}{e^2-1}\int_0^1 2e^{x+t}\ \phi(t)\,dt$

Determine the eigen values and eigen functions (if exists) of the following Fredholm homogeneous integral equations :

5. $\phi(x) = \lambda\int_0^{2\pi} \sin x \cos t\ \phi(t)\,dt$
6. $\phi(x) = \lambda\int_0^{2\pi} \sin x \sin t\ \phi(t)\,dt$
7. $\phi(x) = \lambda\int_0^{\pi} \sin x \sin 2t\ \phi(t)\,dt$
8. $u(x) = \lambda\int_0^1 e^{x+\xi}\ u(\xi)\,d\xi$
9. $u(x) = \lambda\int_0^1 \sin \pi x\ \cos \pi t\ u(t)\,dt$

Answers

1. $\phi(x) = 0$
2. $\phi(x) = c\sin x$
3. $\phi(x) = c$
4. $\phi(x) = c.e^x$
5. eigen values and functions do not exist.
6. $\lambda = \frac{1}{\pi}$, $\phi(x) = c\sin x$
7. eigen values and functions do not exist
8. $\lambda = \frac{2}{e^2-1}$, $u(x) = \frac{2c}{e^2-1}e^x$ or $4(x) = c\ e^x$
9. eigen values and functions do not exist

Determine the eigen values and eigen function (if exists) of the following homogeneous integral equation :

Example 1. $\phi(x) = \lambda\int_0^1 (3x-2)t\ \phi(t)\,dt$

Solution : $\phi(x) = \lambda(3x-2)c$, $c = \int_0^1 t\ \phi(t)\,dt$

$\Rightarrow \quad c = \int_0^1 t\ (\lambda c)(3t-2)\,dt = \lambda c\int_0^1 (3t^2-2t)\,dt$

$\Rightarrow \quad c = \lambda c\left[t^3 - t^2\right]_0^1 = 0 \Rightarrow c = 0 \Rightarrow \phi(x) = 0$ $\quad \therefore$ eigen values and functions do not exist.

Example 2. $u(x)=\lambda\int_0^{2\pi}\sin(x+t)u(t)\,dt$

Solution : $u(x)=\lambda\int_0^{2\pi}[\sin x\cos t+\cos x\sin t]u(t)\,dt$

$$=\lambda\sin x\int_0^{2\pi}\cos t\;u(t)\,dt+\lambda\cos x\int_0^{2\pi}\sin t\;u(t)\,dt$$

$$u(x)=\lambda\sin x\cdot c_1+\lambda\cos x\cdot c_2 \qquad \text{……(1)}$$

$$c_1=\int_0^{2\pi}\cos t\;u(t)\,dt\ , \qquad c_2=\int_0^{2\pi}\sin t\;u(t)\,dt$$

$$c_1=\frac{\lambda c_1}{2}\int_0^{2\pi}\cos t[\lambda c_1\sin t+\lambda c_2\cos t]\,dt$$

$$c_1=\frac{\lambda c_1}{2}\int_0^{2\pi}\sin 2t\;dt+\frac{\lambda c_2}{2}\int_0^{2\pi}(1+\cos 2t)\,dt$$

$$\Rightarrow \qquad c_1=-\frac{\lambda c_1}{4}[\cos 2t]_0^{2\pi}+\frac{\lambda c_2}{2}\left[t+\frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$\Rightarrow \qquad c_1-\lambda\pi c_2=0 \qquad \text{…..(2)}$$

$$c_2=\int_0^{2\pi}\sin t[\lambda c_1\sin t+\lambda c_2\cos t]\,dt$$

$$=\frac{\lambda c_1}{2}\int_0^{2\pi}(1-\cos 2t)\,dt+\frac{\lambda c_2}{2}\int_0^{2\pi}\sin 2t\;dt$$

$$=\frac{\lambda c_1}{2}\left[t-\frac{\sin 2t}{2}\right]_0^{2\pi}-\frac{\lambda c_2}{4}[\cos 2t]_0^{2\pi}$$

$$=\frac{\lambda c_1}{2}(2\pi)=0 \qquad \Rightarrow \qquad \lambda\pi c_1-c_2=0 \qquad \text{…..(3)}$$

$$\Delta=\begin{vmatrix}1 & -\lambda\pi\\ \lambda\pi & -1\end{vmatrix}=-1+\lambda^2\pi^2=0 \qquad \Rightarrow \qquad \lambda^2=\frac{1}{\pi^2} \qquad \lambda=\pm\frac{1}{\pi}$$

For $\lambda=\frac{1}{\pi}$, $c_1=c_2$ and $u(x)=\frac{c_1}{\pi}\sin x+\frac{c_1}{\pi}\cos x=c(\sin x+\cos x)$

For $\lambda=-\frac{1}{\pi}$, $c_1=-c_2$

$$u(x)=\frac{c_2}{\pi}\sin x-\frac{c_2}{\pi}\cos x=c(\sin x-\cos x)$$

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Values of λ	No. of solutions	solution
$\lambda = \frac{1}{\pi}$	infinite	$u(x) = c(\sin x + \cos x)$
$\lambda = -\frac{1}{\pi}$	infinite	$u(x) = c(\sin x - \cos x)$
$\lambda \neq \pm\frac{1}{\pi}$	unique	$u(x) = 0$

EXERCISE 6.2

Determine the eigen values and eigen functions (if exists) of the following homogeneous integral equation :

1. $\phi(x) = \lambda\int_0^1 (2xt - 4x^2)\ \phi(t)\,dt$

2. $\phi(x) = \lambda\int_{-1}^1 (5xt^3 + 4x^2t)\ \phi(t)\,dt$

3. $\phi(x) = \lambda\int_0^1 (t\sqrt{x} - x\sqrt{t})\ \phi(t)\,dt$

4. $\phi(x) = \lambda\int_{-1}^1 (5xt^3 + 4x^2t + 3xt)\ \phi(t)\,dt$

5. $\phi(x) = \lambda\int_{-1}^1 (x\cosh t - t\sinh x)\ \phi(t)\,dt$

6. $\phi(x) = \lambda\int_0^\pi (\cos^2 x \cos 2t + \cos 3x \cos^3 t)\ \phi(t)\,dt$

7. $\phi(x) = \lambda\int_{-1}^1 (x\cosh t - t^2\sinh x)\ \phi(t)\,dt$

8. $\phi(x) = \lambda\int_{-1}^1 (x\cosh t - t\cosh x)\ \phi(t)\,dt$

9. $\phi(x) = \lambda\int_0^1 (45x^2\log t - 9t^2\log x)\phi(t)\,dt$

10. Which of the following is true/false for the Integral equation : $\phi(x) = \lambda\int_0^\pi \cos(x+t)\ \phi(t)\,dt$ .....(1)

(a) For $\lambda = \frac{2}{\pi}$, (1) has infinitely many solutions.

(b) For $\lambda = \frac{-2}{\pi}$, (1) has unique solution.

(c) For $\lambda = 2$, (1) has unique solution.

(d) For $\lambda = -2$, (1) has infinitely many solutions.

(e) For $\lambda = \frac{2}{\pi}$, (1) has at-least one non-trivial solution.

(f) For $\lambda = \frac{-2}{\pi}$, (1) has infinitely many non-trivial solutions.

(g) For $\lambda = \frac{-\pi}{2}$, (1) has only the trivial solution.

(h) For $\lambda = \frac{2}{\pi}$, (1) has only two non-trivial solutions.

Solve the following homogeneous integral equations :

OR

Find the eigen values and corresponding eigen functions of the following integral equations :

11. $\phi(x) = \lambda \int_0^{\frac{\pi}{2}} \cos(x+t)\phi(t)\,dt$

12. $\phi(x) = \lambda \int_0^{\pi} \cos(x+t)\phi(t)\,dt$

13. $\phi(x) = \lambda \int_0^{2\pi} \cos(x+t)\phi(t)\,dt$

14. $\phi(x) = \lambda \int_0^{\frac{\pi}{2}} \cos(x-t)\phi(t)\,dt$

15. $\phi(x) = \lambda \int_0^{\pi} \cos(x-t)\phi(t)\,dt$

16. $\phi(x) = \lambda \int_0^{2\pi} \cos(x-t)\phi(t)\,dt$

17. $\phi(x) = \lambda \int_0^{\frac{\pi}{2}} \sin(x+t)\phi(t)\,dt$

18. $\phi(x) = \lambda \int_0^{\pi} \sin(x+t)\phi(t)\,dt$

19. $\phi(x) = \lambda \int_0^{2\pi} \sin(x+t)\phi(t)\,dt$

20. $\phi(x) = \lambda \int_0^{\frac{\pi}{2}} \sin(x-t)\phi(t)\,dt$

21. $\phi(x) = \lambda \int_0^{\pi} \sin(x-t)\phi(t)\,dt$

22. $\phi(x) = \lambda \int_0^{2\pi} \sin(x-t)\phi(t)\,dt$

Answers

1. $\lambda_1 = \lambda_2 = -3$; $\phi_1(x) = \phi_2(x) = x - 2x^2$

2. $\lambda = \frac{1}{2}$, $\phi(x) = \frac{5}{2}x + \frac{10}{3}x^2$

3. $\lambda = \pm i\sqrt{150}$, there are no real eigen values and no real eigen functions.

4. $\lambda = \frac{1}{4}$, $\phi(x) = x^2 + \frac{3}{2}x$

5. $\lambda = \frac{-e}{2}$, $\phi(x) = \sin hx$

6. $\lambda_1 = \frac{4}{\pi}$, $\phi_1(x) = \cos^2 x$; $\lambda_2 = \frac{8}{\pi}$, $\phi_2(x) = \cos 3x$

7. eigen values and functions do not exist

8. There are no real eigen values and real eigen functions.

9. There are no real eigen values and real eigen functions.

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11. $\phi(x)=\begin{cases} c\left(\cos x-\dfrac{\pi-\sqrt{\pi^2-4}}{2}\sin x\right) & \text{if } \lambda=\dfrac{4}{\sqrt{\pi^2-4}} \\ c\left(\cos x-\dfrac{\pi+\sqrt{\pi^2-4}}{2}\sin x\right) & \text{if } \lambda=\dfrac{-4}{\sqrt{\pi^2-4}} \\ 0 & \text{if } \lambda\neq\pm\dfrac{4}{\sqrt{\pi^2-4}} \end{cases}$

12. $\phi(x)=\begin{cases} c\cos x & \text{if } \lambda=\dfrac{2}{\pi} \\ c\sin x & \text{if } \lambda=\dfrac{-2}{\pi} \\ 0 & \text{if } \lambda\neq\pm\dfrac{2}{\pi} \end{cases}$

13. $\phi(x)=\begin{cases} c\cos x & \text{if } \lambda=\dfrac{1}{\pi} \\ c\sin x & \text{if } \lambda=\dfrac{-1}{\pi} \\ 0 & \text{if } \lambda\neq\pm\dfrac{1}{\pi} \end{cases}$

14. $\phi(x)=\begin{cases} c(\cos x+\sin x) & \text{if } \lambda=\dfrac{4}{\pi+2} \\ c(\cos x-\sin x) & \text{if } \lambda=\dfrac{4}{\pi-2} \\ 0 & \text{if } \lambda\neq\dfrac{4}{\pi\pm2} \end{cases}$

15. $\phi(x)=\begin{cases} c_1\cos x+c_2\sin x & \text{if } \lambda=\dfrac{2}{\pi} \\ 0 & \text{if } \lambda\neq\dfrac{2}{\pi} \end{cases}$

16. $\phi(x)=\begin{cases} c_1\cos x+c_2\sin x & \text{if } \lambda=\dfrac{1}{\pi} \\ 0 & \text{if } \lambda\neq\dfrac{1}{\pi} \end{cases}$

17. $\phi(x)=\begin{cases} c(\cos x+\sin x) & \text{if } \lambda=\dfrac{4}{2+\pi} \\ c(\cos x-\sin x) & \text{if } \lambda=\dfrac{4}{2-\pi} \\ 0 & \text{if } \lambda\neq\dfrac{4}{2\pm\pi} \end{cases}$

18. $\phi(x)=\begin{cases} c(\cos x+\sin x) & \text{if } \lambda=\dfrac{2}{\pi} \\ c(\cos x-\sin x) & \text{if } \lambda=\dfrac{-2}{\pi} \\ 0 & \text{if } \lambda\neq\pm\dfrac{2}{\pi} \end{cases}$

19. $\phi(x)=\begin{cases} c(\cos x+\sin x) & \text{if } \lambda=\dfrac{1}{\pi} \\ c(\cos x-\sin x) & \text{if } \lambda=\dfrac{-1}{\pi} \\ 0 & \text{if } \lambda\neq\pm\dfrac{1}{\pi} \end{cases}$

20. No real eigen values and eigen functions.

In fact $\lambda=\pm\dfrac{4i}{\sqrt{\pi^2-4}}$

21. No real eigen values and eigen functions.

In fact $\lambda=\pm\dfrac{2i}{\pi}$

22. No real eigen values and eigen functions.

In fact $\lambda=\pm\dfrac{1}{\pi}i$

Solve the non-homogeneous Fredholm integral equation with separable kernel :

Example 1. $u(x)=\sin x+\lambda\int_0^{\frac{\pi}{2}}\sin x\cos\xi\, u(\xi)\,d\xi$

Solution : $u(x)=\sin x+\lambda\sin x\cdot c$ ……(1)

$$c = \int_0^{\frac{\pi}{2}} \cos\xi \, u(\xi)\, d\xi$$

$$\Rightarrow \quad c = \int_0^{\frac{\pi}{2}} \cos\xi \left[\sin\xi + \lambda c \sin\xi\right] d\xi$$

$$\Rightarrow \quad c = \int_0^{\frac{\pi}{2}} \left(\frac{1+\lambda c}{2}\right)\sin 2\xi \, d\xi \qquad \Rightarrow \quad c = \left(\frac{1+\lambda c}{2}\right)\left(-\frac{\cos 2\xi}{2}\right)_0^{\frac{\pi}{2}}$$

$$\Rightarrow \quad c = -\left(\frac{1+\lambda c}{4}\right)\left[-1-1\right] \qquad \Rightarrow \quad c = \frac{1+\lambda c}{2}$$

$$\Rightarrow \quad c = \left(1-\frac{\lambda}{2}\right) = \frac{1}{2} \qquad \ldots\ldots(2)$$

$$c = \frac{1}{2-\lambda}, \quad \lambda \neq 2 \qquad \Rightarrow \quad u(x) = \sin x\left(1+\frac{\lambda}{2-\lambda}\right) = \frac{2\sin x}{2-\lambda}$$

Values of λ	No. of solutions	solution
$\lambda = 2$	no solution	X
$\lambda \neq 2$	unique solution	$u(x) = \dfrac{2}{2-\lambda}\sin x$

EXERCISE 6.3

Solve the following non-homogeneous Fredholm integral equations with separable kernel :

1. $y(x) = \tan x + \int_{-1}^{1} e^{\sin^{-1} x}\, y(t)\, dt$

2. $\phi(x) = \sec x \tan x - \lambda \int_0^1 \phi(t)\, dt$

3. $y(x) = \frac{1}{\sqrt{1-x^2}} + \lambda \int_0^1 \cos^{-1} t \; y(t)\, dt$

4. $\phi(x) = \sec^2 x + \lambda \int_0^1 \phi(t)\, dt$

5. $\phi(x) - \lambda \int_{-\pi/4}^{\pi/4} \tan t \;\phi(t)\, dt = \cot x$

6. $\phi(x) - \lambda \int_0^1 (x\log t - t\log x)\; \phi(t)\, dt = \frac{6}{5}(1-4x)$

7. $\phi(x) - \lambda \int_0^{2\pi} |\pi - t| \sin x \;\phi(t)\, dt = x$

8. $\phi(x) = x + \lambda \int_0^1 (1+x+t)\; \phi(t)\, dt$

9. $\phi(x) = x + \lambda \int_0^{\pi} (1+\sin x \sin t)\, \phi(t)\, dt$

10. $\phi(x) = x + \lambda \int_0^1 \left(4xt - x^3\right) \phi(t)\, dt$

11. $\phi(x) - \lambda \int_0^1 \left(4xt - x^2\right) \phi(t)\, dt = x$

12. $u(x) = e^x + \lambda \int_0^1 2e^{x+t}\; u(t)\, dt$

13. $\phi(x) = \cos x + \lambda \int_0^{\pi} \sin x \,\phi(t)\, dt$

14. $\phi(x) = 2x - \pi + 4\int_0^{\pi/2} \sin^2 x \,\phi(t)\, dt$

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15. $u(x)=x+\lambda\int_0^1\left(x\xi^2+x^2\xi\right)u(\xi)\,d\xi$

16. $y(x)=1+\int_0^1\left(1+e^{x+t}\right)\,y(t)\,dt$

17. $\phi(x)=(1+x)^2+\int_{-1}^1\left(xt+x^2t^2\right)\phi(t)\,dt$

Solve the following non-homogeneous integral equations :

18. $\phi(x)=1+\lambda\int_0^{\frac{\pi}{2}}\cos(x+t)\phi(t)\,dt$

19. $\phi(x)=x+\lambda\int_0^{\frac{\pi}{2}}\cos(x+t)\phi(t)\,dt$

20. $\phi(x)=1+\lambda\int_0^{\pi}\cos(x+t)\phi(t)\,dt$

21. $\phi(x)=x+\lambda\int_0^{\pi}\cos(x+t)\phi(t)\,dt$

22. $\phi(x)=1+\lambda\int_0^{2\pi}\cos(x+t)\phi(t)\,dt$

23. $\phi(x)=x+\lambda\int_0^{2\pi}\cos(x+t)\phi(t)\,dt$

24. $\phi(x)=1+\lambda\int_0^{\frac{\pi}{2}}\cos(x-t)\phi(t)\,dt$

25. $\phi(x)=x+\lambda\int_0^{\frac{\pi}{2}}\cos(x-t)\phi(t)\,dt$

26. $\phi(x)=1+\lambda\int_0^{\pi}\cos(x-t)\phi(t)\,dt$

27. $\phi(x)=x+\lambda\int_0^{\pi}\cos(x-t)\phi(t)\,dt$

28. $\phi(x)=1+\lambda\int_0^{2\pi}\cos(x-t)\phi(t)\,dt$

29. $\phi(x)=x+\lambda\int_0^{2\pi}\cos(x-t)\phi(t)\,dt$

30. $\phi(x)=1+\lambda\int_0^{\frac{\pi}{2}}\sin(x+t)\phi(t)\,dt$

31. $\phi(x)=x+\lambda\int_0^{\frac{\pi}{2}}\sin(x+t)\phi(t)\,dt$

32. $\phi(x)=1+\lambda\int_0^{\pi}\sin(x+t)\phi(t)\,dt$

33. $\phi(x)=x+\lambda\int_0^{\pi}\sin(x+t)\phi(t)\,dt$

34. $\phi(x)=1+\lambda\int_0^{2\pi}\sin(x+t)\phi(t)\,dt$

35. $\phi(x)=x+\lambda\int_0^{2\pi}\sin(x+t)\phi(t)\,dt$

Solve the following non-homogeneous Fredholm integral equations with separable kernel :

36. $\phi(x)=\cos x+\lambda\int_0^{\pi}\sin(x-t)\,\phi(t)\,dt$

37. $\phi(x)=\cos 2x+\lambda\int_0^{\pi}\cos(x+\xi)\,\phi(\xi)\,d\xi$

Answers

1. $y(x)=\tan x$

2. $\phi(x)=\begin{cases}\sec x\,\tan x-\dfrac{\lambda}{1+\lambda}\sec 1, & \lambda\neq -1\\ \text{No Solution}, & \lambda=-1\end{cases}$

3. $y(x)=\begin{cases}\dfrac{1}{\sqrt{1-x^2}}-\dfrac{\pi^2}{8(\lambda-1)}, & \lambda\neq 1\\ \text{No Solution}, & \lambda=1\end{cases}$

4. $\phi(x)=\begin{cases}\sec^2 x+\dfrac{\lambda}{1-\lambda}\tan 1, & \lambda\neq 1\\ \text{No Solution}, & \lambda=1\end{cases}$

5. $\phi(x) = \cot x + \dfrac{\pi\lambda}{2}$

For all λ

6. $\dfrac{6}{5}(1-4x) + \dfrac{2\lambda^2 x + \left(\lambda + \lambda^2/4\right) l_n x}{1 + (29/48)\lambda^2}$

for Det=0 $\begin{pmatrix}\text{No Real} \\ \text{Eigen Values}\end{pmatrix}$

7. $\phi(x) = x + \lambda\pi^3 \sin x$

8. $\phi(x) = x + \dfrac{\lambda}{12 - 24\lambda - \lambda^2}\left[10 + (6+\lambda)x\right]$

for Det=0 $\quad \lambda = \dfrac{-24 \pm \sqrt{624}}{2}$

9. $\phi(x) = x + \dfrac{\lambda}{(1-\lambda\pi)\left[1 - \frac{1}{2}\lambda\pi\right] - 4\lambda^2}\left[2\lambda\pi + \frac{1}{2}\pi^2\left[1 - \frac{1}{2}\lambda\pi\right] + \pi\sin x\right]$

10. $\phi(x) = \dfrac{15(4+\lambda)x - 30\lambda x^3}{60 - 65\lambda + 4\lambda^2}$

11. $\phi(x) = \dfrac{6(3+\lambda)x - 9\lambda x^2}{18 - 18\lambda + \lambda^2}$

12. $u(x) = \begin{cases} \dfrac{e^x}{1 - \lambda\left(e^2 - 1\right)}, & \lambda \neq \dfrac{1}{e^2 - 1} \\ \text{No Solution}, & \lambda = \dfrac{1}{e^2 - 1} \end{cases}$

13. $\phi(x) = \begin{cases} \cos x, & \lambda \neq \dfrac{1}{2} \\ \cos x + c\sin x, & \lambda = \dfrac{1}{2} \end{cases}$

14. $\phi(x) = 2x - \pi + \dfrac{\pi^2 \sin^2 x}{\pi - 1}$

15. $u(x) = \dfrac{(240 - 60\lambda)x + 80\lambda x^2}{240 - 120\lambda - \lambda^2}$

16. $y(x) = \dfrac{e^2 - 3 - 2e^x(e-1)}{2(e-1)^2}$

17. $\phi(x) = 1 + 6x + \dfrac{25}{9}x^2$

36. $\phi(x) = \dfrac{4\cos x + 2\pi\lambda \sin x}{4 + \lambda^2\pi^2}$

37. $\phi(x) = \begin{cases} \cos 2x + \dfrac{4\lambda}{3(2+\lambda\pi)}\sin x, & \lambda \neq \pm\dfrac{2}{\pi} \\ \cos 2x + c\cos x + \dfrac{2}{3\pi}\sin x, & \lambda = \dfrac{2}{\pi} \\ \text{No Solution}, & \lambda = \dfrac{-2}{\pi} \end{cases}$

EXERCISE 6.4

Solve the following non-homogeneous Fredholm integral equations with separable kernel :

1. $u(x) = f(x) + \lambda\int_0^1 x\xi\, u(\xi)\, d\xi$

2. $u(x) = f(x) + \lambda\int_0^{2\pi} (\sin x \cos t)\, u(t)\, dt$

3. $\phi(x) = f(x) + \lambda\int_0^1 (x+t)\, \phi(t)\, dt$

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4. $y(x) = f(x) + \lambda\int_{-1}^{1}\left(xt + x^2t^2\right) y(t)\,dt$.

5. Show that the Integral equation $u(x) = f(x) + \frac{1}{\pi}\int_0^{2\pi}\sin(x+t)\,u(t)\,dt$ possesses no solution for $f(x) = x$, but that it possesses infinitely many solutions when $f(x) = 1$.

6. Solve the Integral equation and discuss all it's possible cases $\phi(x) = f(x) + \lambda\int_0^1 (1-3xt)\,\phi(t)\,dt$

i.e., (i) When $f(x) = 0$ (ii) When $f(x) \neq 0$

7. $y(x) = F(x) + \lambda\int_0^{2\pi}\cos(x+t)\; y(t)\,dt$

(a) Determine the characteristic values of λ and the characteristic functions.

(b) Obtain the general solution (when it exists) if $F(x) = \sin x$, considering all possible cases.

Answers

1. $u(x) = f(x) + \frac{3\lambda x}{3-\lambda}\int_0^1 \xi\; f(\xi)\,d\xi$

2. $u(x) = f(x) + \lambda\int_0^{2\pi}(\cos t\,\sin x)\; f(t)\,dt$

3. $\phi(x) = f(x) + \lambda\int_0^1 \frac{6(\lambda-2)(x+t) - 12\lambda\,xt - 4\lambda}{\lambda^2 + 12\lambda - 12}\; f(t)\,dt$

4. $y(x) = f(x) + \lambda\int_{-1}^{1}\left[\frac{3xt}{3-2\lambda} + \frac{5x^2t^2}{5-2\lambda}\right] f(t)\,dt$

5. When $f(x) = 1$, $u(x) = 1 + c(\sin x + \cos x)$

6. (i) When $f(x) = 0$, infinite solutions if $\lambda = \pm 2$ and unique solution if $\lambda \neq \pm 2$

When $f(x) \neq 0$ and $\lambda \neq \pm 2$

(ii) $$(x) = f(x) + \lambda\int_0^1\left(\frac{-6x\lambda\xi - 3\lambda\xi + 6x\lambda^2\xi - 3x\lambda^2 + 2\lambda + 2}{2\left(1 - \frac{\lambda^2}{4}\right)}\right) f(\xi)\,d\xi$$

7. (a) $\lambda_1 = \frac{1}{\pi}, y_1(x) = \cos x$; $\lambda_2 = \frac{-1}{\pi}, y_2(x) = \sin x$

(b) $y(x) = \frac{\sin x}{1+\pi\lambda}$ if $\lambda \neq \pm\frac{1}{\pi}$; $y(x) = \frac{1}{2}\sin x + A\cos x$, A arbitrary, if $\lambda = \frac{1}{\pi}$; No solution if $\lambda = -\frac{1}{\pi}$

6.5 Solution of some special type of Volterra integral equation :

1. $\int_a^x (x-t)^n y(t)\,dt = f(x), \quad n = 1, 2, ...$

It is assumed that the right-hand of the equation satisfies the conditions

$f(a) = f'(a) = ... = f^{(n)}(a) = 0$

Solution : $y(x) = \frac{1}{n!} f^{(n+1)}(x)$

Example : For $f(x) = Ax^m$, where m is a positive integer, $m > n$, the solution has the form

$$y(x) = \frac{Am!}{n!(m-n-1)!} x^{m-n-1}$$

(i) $\int_a^x y(t)\,dt = f(x)$, Solution : $y(x) = f'(x)$

(ii) $\int_a^x (x-t)y(t)\,dt = f(x)$, Solution : $y(x) = f''(x)$

(iii) $\int_a^x (x-t)^2 y(t)\,dt = f(x)$, Solution : $y(x) = \frac{1}{2} f'''(x)$ $\quad f(a) = f'(a) = f''(a) = 0$

(iv) $\int_a^x (x-t)^3 y(t)\,dt = f(x)$, Solution : $y(x) = \frac{1}{6} f''''(x)$ $\quad f(a) = f'(a) = f''(a) = f'''(a) = 0$

2. $\int_a^x (x^n - t^n) y(t)\,dt = f(x)$, Solution : $y(x) = \frac{1}{n}\frac{d}{dx}\left[\frac{f'(x)}{x^{n-1}}\right]$ $\quad f(a) = f'(a) = 0, \quad n = 1, 2, ...$

(i) $\int_a^x (x-t)y(t)\,dt = f(x)$ Solution : $y(x) = f''(x)$

(ii) $\int_a^x (x^2 - t^2) y(t)\,dt = f(x)$, Solution : $y(x) = \frac{1}{2x^2}\left[xf''(x) - f'(x)\right]$ $\quad f(a) = f'(a) = 0$

(iii) $\int_a^x (x^3 - t^3) y(t)\,dt = f(x)$, Solution : $y(x) = \frac{1}{3x^3}\left[xf'''(x) - 2f'(x)\right]$ $\quad f(a) = f'(a) = 0$

3. $\int_a^x \left(t^n x^{n+1} - x^n t^{n+1}\right) y(t)\,dt = f(x), \quad n = 2, 3, ...$ Solution : $y(x) = \frac{1}{x^n}\frac{d^2}{dx^2}\left[\frac{f(x)}{x^n}\right]$

(i) $\int_a^x \left(x^2 t - x t^2\right) y(t)\,dt = f(x), \qquad f(a) = f'(a) = 0 \qquad$ Solution : $y(x) = \frac{1}{x}\frac{d^2}{dx^2}\left[\frac{1}{x} f(x)\right]$

4. (i) $\int_a^x e^{\lambda(x-t)} y(t)\,dt = f(x), \qquad$ Solution : $y(x) = f'(x) - \lambda f(x)$

Example : In the special case $a = 0$ and $f(x) = Ax$, the solution has the form $y(x) = A(1 - \lambda x)$.

(ii) $\int_a^x e^{\lambda x + \beta t} y(t)\,dt = f(x), \qquad$ Solution : $y(x) = e^{-(\lambda+\beta)x}\left[f'(x) - \lambda f(x)\right]$

Example : In the special case $a = 0$ and $f(x) = A\sin(\gamma x)$, the solution has the form

$y(x) = Ae^{-(\lambda+\beta)x} \times \left[\gamma\cos(\gamma x) - \lambda\sin(\gamma x)\right]$

(iii) $\int_a^x \left(e^{\lambda x} - e^{\lambda t}\right) y(t)\,dt = f(x), \qquad f(a) = f'(a) = 0 \qquad$ Solution : $y(x) = e^{-\lambda x}\left[\frac{1}{\lambda} f''(x) - f'(x)\right]$

(iv) $\int_a^x \left(e^{\lambda(x-t)} - e^{\mu(x-t)}\right) y(t)\,dt = f(x), \quad f(a) = f'(a) = 0$

Solution : $y(x) = \frac{1}{\lambda - \mu}\left[f'' - (\lambda + \mu) f' + \lambda\mu f\right], \qquad f = f(x)$

5. $\int_a^x (x-t)^n e^{\lambda(x-t)} y(t)\,dt = f(x), \qquad n = 1, 2, ...$

It is assumed that $f(a) = f'(a) = ... = f^{(n)}(a) = 0$

Solution : $y(x) = \frac{1}{n!} e^{\lambda x} \frac{d^{n+1}}{dx^{n+1}}\left[e^{-\lambda x} f(x)\right]$

(i) $\int_a^x (x-t)^2 e^{\lambda(x-t)} y(t)\,dt = f(x), \qquad f(a) = f'(a) = f''(a) = 0$

Solution : $y(x) = \frac{1}{2}\left[f'''(x) - 3\lambda f''(x) + 3\lambda^2 f'(x) - \lambda^3 f(x)\right]$

6.

(i) $\int_a^x \cosh\left[\lambda(x-t)\right] y(t)\,dt = f(x), \qquad$ Solution : $y(x) = f'(x) - \lambda^2 \int_a^x f(x)\,dx$

(ii) $\int_a^x [\cosh(\lambda x) - \cosh(\lambda t)] y(t) dt = f(x)$, Solution : $y(x) = \frac{1}{\lambda} \frac{d}{dx} \left[\frac{f'(x)}{\sinh(\lambda x)} \right]$

(iii) $\int_a^x \sinh[\lambda(x-t)] y(t) dt = f(x)$, Solution : $y(x) = \frac{1}{\lambda} f''(x) - \lambda f(x)$ $f(a) = f'(a) = 0$

(iv) $\int_a^x [\sinh(\lambda x) - \sinh(\lambda t)] y(t) dt = f(x)$, Solution : $y(x) = \frac{1}{\lambda} \frac{d}{dx} \left[\frac{f'(x)}{\cosh(\lambda x)} \right]$ $f(a) = f'(a) = 0$

7. (i) $\int_a^x (\ln x - \ln t) y(t) dt = f(x)$ Solution : $y(x) = x\, f''(x) + f'(x)$

(ii) $\int_a^x \left(\frac{x+b}{t+b} \right) y(t) dt = f(x)$, Solution : $y(x) = (x+b) f''(x) + f'(x)$

8. (i) $\int_a^x \cos[\lambda(x-t)] y(t) dt = f(x)$, Solution : $y(x) = f'(x) + \lambda^2 \int_a^x f(x) dx$

(ii) $\int_a^x \{\cos[\lambda(x-t)] - 1\} y(t) dt = f(x)$, $f(a) = f'(a) = f''(x) = 0$

Solution : $y(x) = -\frac{1}{\lambda^2} f'''(x) - f'(x)$

(iii) $\int_a^x \sin[\lambda(x-t)] y(t) dt = f(x)$, $f(a) = f'(a) = 0$ Solution : $y(x) = \frac{1}{\lambda} f''(x) + \lambda f(x)$

7. Solution of Fredholm Integral equation with the help of Resolvent Kernel

Formula :

1. Iterated Kernel : $K_1(x,t) = K(x,t)$

$$K_{n+1}(x,t) = \int_a^b K(x,z)\, K_n(z,t) dz .$$

2. Resolvent Kernel : $R(x,t;\lambda) = K_1 + \lambda K_2 + \lambda^2 K_3 + ...\ = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x,t)$

3. Solution : $\phi(x) = f(x) + \lambda \int_a^b R(x,t;\lambda)\, f(t) dt$

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EXERCISE 7.1

1. Find the iterated kernels for the following kernels:

   (i) $K(x,t)=\sin(x-2t)$, $0\le x\le 2\pi$, $0\le t\le 2\pi$ (ii) $K(x,t)=e^x\cos t$; $a=0$, $b=\pi$

   (iii) $K(x,t)=x+\sin t$; $a=-\pi$, $b=\pi$ (iv) $K(x,t)=x-t$; $a=0$, $b=1$

2. Determine the Resolvent kernel or Reciprocal kernel for the following kernels:

   (i) $K(x,t)=e^{x+t}$; $a=0$, $b=1$ (ii) $K(x,t)=(1+x)(1-t)$; $a=-1$, $b=1$

3. Solve the following Fredholm integral equations with the help of Resolvent kernel:

   (i) $y(x)=x+\int_0^{1/2} y(t)\,dt$ (ii) $y(x)=e^x-\frac{1}{2}e+\frac{1}{2}+\frac{1}{2}\int_0^1 y(t)\,dt$ (iii) $y(x)=\frac{5x}{6}+\frac{1}{2}\int_0^1 xt\,y(t)\,dt$

   (iv) $y(x)=x+\lambda\int_0^1 xt\,y(t)\,dt$ (v) $y(x)=\sin x-\frac{x}{4}+\frac{1}{4}\int_0^{\pi/2} xt\,y(t)\,dt$ (vi) $y(t)=f(x)+\lambda\int_0^1 e^{x-t}\,y(t)\,dt$

   (vii) $y(x)=\frac{3}{2}e^x-\frac{1}{2}xe^x-\frac{1}{2}+\frac{1}{2}\int_0^1 t\,y(t)\,dt$ (viii) $y(x)=1+\lambda\int_0^{\pi}\sin(x+t)\,y(t)\,dt$

   (ix) $y(x)=1+\lambda\int_0^1(1-3xt)\,y(t)\,dt$

Answers

1. (i) $K_n(x,t)=0$ for $n=2,3,4,\ldots\ldots\ldots$ (ii) $K_n(x,t)=(-1)^{n-1}\left[\frac{1+e^{\pi}}{2}\right]^{n-1}e^x\cos t$, $n=1,2,3\ldots\ldots$

   (iii) if $n=2m-1$, $K_{2m-1}(x,t)=(2\pi)^{2m-2}(x+\sin t)$, $m=1,2,3,\ldots$

   if $n=2m$, $K_{2m}(x,t)=(2\pi)^{2m-1}(1+x\sin t)$, $m=1,2,3,\ldots$

   (iv) if $n=2m-1$, then $K_{2m-1}(x,t)=\frac{(-1)^{m-1}}{12^{m-1}}(x-t)$, $m=1,2,3,\ldots$

   if $n=2m$, then $K_{2m}(x,t)=\frac{(-1)^{m-1}}{12^{m-1}}\left[\frac{x+t}{2}-\frac{1}{3}-xt\right]$, $m=1,2,3,\ldots$

2. (i) $R(x,t;\lambda)=\frac{2e^{x+t}}{2-\lambda(e^2-1)}$, provided $|\lambda|<\frac{2}{e^2-1}$ (ii) $R(x,t;\lambda)=\frac{3(1+x)(1-t)}{3-4\lambda}$, provided $|\lambda|<\frac{3}{4}$

3. (i) $y(x)=x+\frac{1}{4}$ (ii) $y(x)=e^x$

(iii) $y(x) = x$

(iv) $y(x) = \dfrac{3x}{3-\lambda}$ where $|\lambda| < 3$

(v) $y(x) = \sin x$

(vi) $y(x) = f(x) + \dfrac{\lambda}{1-\lambda}\int_0^1 e^{x-t} f(t)\,dt$, where $|\lambda| < 1$

(vii) $y(x) = \dfrac{3}{2}e^x - \dfrac{1}{2}xe^x - \dfrac{e}{3} + 1$

(viii) $y(x) = 1 + \dfrac{4\lambda}{4-\lambda^2\pi^2}(2\cos x + \lambda\pi\sin x)$, where $|\lambda| < \dfrac{2}{\pi}$

(ix) $y(x) = \dfrac{4+2\lambda(2-3x)}{4-\lambda^2}$, $|\lambda| < 2$

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Differentiation Formulas

Procedural rules :

1. Constant multiple rule $(cf)' = cf'$
2. Sum rule $(f+g)' = f' + g'$
3. Difference rule $(f-g)' = f' - g'$
4. Linearity rule $(af+bg)' = af' + bg'$
5. Product rule $(fg)' = fg' + f'g$
6. Quotient rule $\left(\dfrac{f}{g}\right)' = \dfrac{gf' - fg'}{g^2}$
7. Chain rule $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

Basic formulas :

1. Extended power rule $\dfrac{d}{dx}u^n = nu^{n-1}\dfrac{du}{dx}$

2. Trigonometric rules

(i) $\dfrac{d}{dx}\sin u = \cos u\dfrac{du}{dx}$ (ii) $\dfrac{d}{dx}\cos u = -\sin u\dfrac{du}{dx}$

(iii) $\dfrac{d}{dx}\tan u = \sec^2 u\dfrac{du}{dx}$ (iv) $\dfrac{d}{dx}\cot u = \csc^2 u\dfrac{du}{dx}$

(v) $\dfrac{d}{dx}\sec u = \sec u\tan u\dfrac{du}{dx}$ (vi) $\dfrac{d}{dx}\csc u = -\csc u\cot u\dfrac{du}{dx}$

3. Inverse trigonometric rules

(i) $\dfrac{d}{dx}\sin^{-1}u = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$ (ii) $\dfrac{d}{dx}\cos^{-1}u = \dfrac{-1}{\sqrt{1-u^2}}\dfrac{du}{dx}$

(iii) $\dfrac{d}{dx}\tan^{-1}u = \dfrac{1}{1+u^2}\dfrac{du}{dx}$ (iv) $\dfrac{d}{dx}\cot^{-1}u = \dfrac{-1}{1+u^2}\dfrac{du}{dx}$

(v) $\dfrac{d}{dx}\sec^{-1}u = \dfrac{1}{|u|\sqrt{u^2-1}}\dfrac{du}{dx}$ (vi) $\dfrac{d}{dx}\csc^{-1}u = \dfrac{-1}{|u|\sqrt{u^2-1}}\dfrac{du}{dx}$

4. Logarithmic rules

(i) $\dfrac{d}{dx}\ln u = \dfrac{1}{u}\dfrac{du}{dx}$ (ii) $\dfrac{d}{dx}\log_b u = \dfrac{\log_b e}{u}\dfrac{du}{dx} = \dfrac{1}{u\ln b}\dfrac{du}{dx}$

5. Exponential rules

(i) $\dfrac{d}{dx}e^u = e^u\dfrac{du}{dx}$ (ii) $\dfrac{d}{dx}b^u = b^u\ln b\dfrac{du}{dx}$

6. Hyperbolic rules

(i) $\frac{d}{dx}\sinh u = \cosh u \frac{du}{dx}$ (ii) $\frac{d}{dx}\cosh u = -\sinh u \frac{du}{dx}$

(iii) $\frac{d}{dx}\tanh u = \operatorname{sech}^2 u \frac{du}{dx}$ (iv) $\frac{d}{dx}\coth u = \operatorname{csch}^2 u \frac{du}{dx}$

(v) $\frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u \tanh u \frac{du}{dx}$ (vi) $\frac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u \coth u \frac{du}{dx}$

7. Inverse hyperbolic rules

(i) $\frac{d}{dx}\sinh^{-1} u = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$ (ii) $\frac{d}{dx}\cosh^{-1} u = \frac{-1}{\sqrt{u^2-1}}\frac{du}{dx}$

(iii) $\frac{d}{dx}\tanh^{-1} u = \frac{1}{1-u^2}\frac{du}{dx}$ (iv) $\frac{d}{dx}\coth^{-1} u = \frac{1}{1-u^2}\frac{du}{dx}$

(v) $\frac{d}{dx}\operatorname{sech}^{-1} u = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}$ (vi) $\frac{d}{dx}\operatorname{csch}^{-1} u = \frac{-1}{|u|\sqrt{1+u^2}}\frac{du}{dx}$

Integration Formulas

Procedural rules :

1. Constant multiple rule $\int cf(u)\,du = c\int f(u)\,du$
2. Sum rule $\int [f(u)+g(u)]\,du = \int f(u)\,du + \int g(u)\,du$
3. Difference rule $\int [f(u)-g(u)]\,du = \int f(u)\,du - \int g(u)\,du$
4. Linearity rule $\int [af(u)+bg(u)]\,du = a\int f(u)\,du + b\int g(u)\,du$

Basic formulas :

1. Constant rule $\int 0\,du = C$
2. Power rule (i) $\int u^n\,du = \frac{u^{n+1}}{n+1}+C;\ n \neq -1$

   (ii) $\int u^{-1}\,du = \ln|u| + C$
3. Exponential rule (i) $\int e^u\,du = e^u + C$

   (ii) $\int a^u\,du = \frac{a^u}{\ln a}+C \quad a>0,\ a\neq 1$
4. Logarithmic rule $\int \ln u\,du = u\ln u - u + C,\ u>0$
5. Trigonometric rules (i) $\int \sin u\,du = -\cos u + C$ (ii) $\int \cos u\,du = \sin u + C$

(iii) $\int \tan u\, du = -\ln|\cos u| + C$ $= \ln|\sec u| + C$

(iv) $\int \cot u\, du = \ln|\sin u| + C$

(v) $\int \sec u\, du = \ln|\sec u + \tan u| + C$

(vi) $\int \csc u\, du = -\ln|\csc u + \cot u| + C$

(vii) $\int \sec^2 u\, du = \tan u + C$

(viii) $\int \csc^2 u\, du = -\cot u + C$

(ix) $\int \sec u \tan u\, du = \sec u + C$

(x) $\int \csc u \cot u\, du = -\csc u + C$

6. Cosine squared formula $\int \cos^2 u\, du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$

7. Sine squared formula $\int \sin^2 u\, du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

8. Hyperbolic rules

(i) $\int \sinh u\, du = \cosh u + C$

(ii) $\int \cosh u\, du = \sinh u + C$

(iii) $\int \tanh u\, du = \ln(\cosh u) + C$

(iv) $\int \coth u\, du = \ln|\sinh u| + C$

9. Inverse rules

(i) $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C$

(ii) $\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\frac{u}{a} + C$

(iii) $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

(iv) $$\int \frac{du}{a^2 - u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\frac{u}{a} + C & \text{if } \left|\frac{u}{a}\right| < 1 \\ \frac{1}{a}\coth^{-1}\frac{u}{a} + C & \text{if } \left|\frac{u}{a}\right| > 1 \end{cases}$$

(v) $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{|u|}{a} + C$

(vi) $\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left|\frac{u}{a}\right| + C = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C$

(vii) $\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\frac{u}{a} + C = \ln\left(\sqrt{a^2 + u^2} + u\right) + C$

(viii) $\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2 + u^2} + a}{u}\right| + C$

Table of formulae :

Sr. No.		**Trigonometric Functions**	**Hyperbolic Functions**
1.	(i)	$\sin^2 x + \cos^2 x = 1$	$\cosh^2 x - \sinh^2 x = 1$
	(ii)	$\sec^2 x - \tan^2 x = 1$	$\operatorname{sech}^2 x + \tanh^2 x = 1$
	(iii)	$\operatorname{cosec}^2 x - \cot^2 x = 1$	$\coth^2 x - \operatorname{cosech}^2 x = 1$
2.	(i)	$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$	$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
	(ii)	$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$	$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
	(iii)	$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$	$\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
3.	(i)	$\sin(-x) = -\sin x$	$\sinh(-x) = -\sinh x$
	(ii)	$\cos(-x) = \cos x$	$\cosh(-x) = \cosh x$
	(iii)	$\tan(-x) = -\tan x$	$\tanh(-x) = -\tanh x$
4.	(i)	$\sin 2x = 2\sin x \cos x$	$\sinh 2x = 2\sinh x \cosh x$
	(ii)	$\cos 2x = \cos^2 x - \sin^2 x$	$\cosh 2x = \cosh^2 x + \sinh^2 x$
	(iii)	$\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$	$\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$
5.	(i)	$\sin 3x = 3\sin x - 4\sin^3 x$	$\sinh 3x = 3\sinh x + 4\sinh^3 x$
	(ii)	$\cos 3x = 4\cos^3 x - 3\cos x$	$\cosh 3x = 4\cosh^3 x - 3\cosh x$
	(iii)	$\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$	$\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$
6.	(i)	$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$2\sinh A \cosh B = \sinh(A+B) + \sinh(A-B)$
	(ii)	$2\cos A \sin B = \sin(A+B) - \sin(A-B)$	$2\cosh A \sinh B = \sinh(A+B) - \sinh(A-B)$
	(iii)	$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	$2\cosh A \cosh B = \cosh(A+B) + \cosh(A-B)$
	(iv)	$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	$2\sinh A \sinh B = \cosh(A+B) - \cosh(A-B)$
7.	(i)	$\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$	$\sinh C + \sinh D = 2\sinh\dfrac{C+D}{2}\cosh\dfrac{C-D}{2}$
	(ii)	$\sin C - \sin D = 2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$	$\sinh C - \sinh D = 2\cosh\dfrac{C+D}{2}\sinh\dfrac{C-D}{2}$
	(iii)	$\cos C + \cos D = 2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$	$\cosh C + \cosh D = 2\cosh\dfrac{C+D}{2}\cosh\dfrac{C-D}{2}$
	(iv)	$\cos C - \cos D = 2\sin\dfrac{C+D}{2}\sin\dfrac{D-C}{2}$	$\cosh C - \cosh D = 2\sinh\dfrac{C+D}{2}\sinh\dfrac{C-D}{2}$

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-------------------------- S C Q ----------------------

1. Integral equation for boundary value problem $y''-5y'+6y=0$, given that $y(0)=0,\ y'(0)=-1$ is
   1. $y(x)=\int_0^x (5-6x+6t)\,y(t)\,dt$
   2. $y(x)=-\int_0^5 (5-6x+6t)\,y(t)\,dt$
   3. $y(x)=6x-5+\int_0^x (5-6x+6t)\,y(t)\,dt$
   4. $y(x)=6x+5+\int_0^x (5-6x+6t)\,y(t)\,dt$

2. The differential equation corresponding to integral equation $y(x)=\int_0^x (x+t)\,y(t)\,dt+1$ is
   1. $y''(x)-y(x)+3\sin x=0, y(0)=3, y'(0)=0$
   2. $y''(x)+6y(x)=0, y(0)=4,$ $y'(0)=-3,\ y''(0)=2$
   3. $y''(x)-y'(x)+4\cos x=0,$ $y(0)=-2, y'(0)=1$
   4. $y''(x)-2xy'(x)-3y(x)=0,$ $y(0)=1, y'(0)=0$

3. The solution of integral equation $\int_0^x e^{x-t}\phi(t)\,dt=\sinh x$ is
   1. $\phi(x)=e^{-x}$
   2. $\phi(x)=e^{x}$
   3. $\phi(x)=\sinh x$
   4. $\phi(x)=\cosh x$

4. For the kernel $K(x,t)=(1+x)(1-t);\ a=-1, b=0$. The third iterated kernel $k_3(x,t)$ is
   1. $(1+x)(1-t)$
   2. $\frac{2}{3}(1+x)(1-t)$
   3. $\left(\frac{2}{3}\right)^2(1+x)(1-t)$
   4. None of the above

5. Which of the following is solution of integral equation $\int_0^x \frac{\phi(t)}{\sqrt{x-t}}\,dt=\sqrt{x}$?
   1. $\phi(x)=\frac{1}{2}$
   2. $\phi(x)=-\frac{1}{2}$
   3. $\phi(x)=1$
   4. $\phi(x)=-1$

6. The initial value problem corresponding to the integral equation $y(x)=1+\int_0^x y(t)\,dt$ is
   1. $y'-y=0,\ y(0)=1$
   2. $y'+y=0,\ y(0)=0$
   3. $y'-y=0,\ y(0)=0$
   4. $y'+y=0,\ y(0)=1$

   (GATE 2001)

7. The integral equation $y(x)=\int_0^x (x-t)\,y(t)\,dt-x\int_0^1 (1-t)\,y(t)\,dt$ is equivalent to
   1. $y''-y=0,\ y(0)=0,\ y(1)=0$

2. $y''-y=0,\ y(0)=0,\ y'(0)=0$
3. $y''+y=0,\ y(0)=0,\ y(1)=0$
4. $y''+y=0,\ y(0)=0,\ y'(0)=0$

(GATE 2003)

8. The integral equation

$$y(x)=\lambda\int_0^{2\pi}\sin(x+t)\,y(t)\,dt \text{ has}$$

1. no solution for any value of λ
2. unique solution for every value of λ
3. infinitely many solutions for only one value of λ
4. infinitely many solutions for two values of λ

(GATE 2003)

9. The value of λ for which the integral equation $y(x)=\lambda\int_0^1(6x-t)\,y(t)\,dt$ has a non-trivial solution, are given by the roots of the equation

1. $(3\lambda-1)(2+\lambda)-\lambda^2=0$
2. $(3\lambda-1)(2+\lambda)+2=0$
3. $(3\lambda-1)(2+\lambda)-4\lambda^2=0$
4. $(3\lambda-1)(2+\lambda)+\lambda^3=0$

(GATE 2004)

10. Given that, the eigen values of the integral equation $y(x)=\lambda\int_0^{2\pi}\cos(x+t)\,y(t)\,dt$ are $\frac{1}{\pi}$ and $-\frac{1}{\pi}$ with respective eigen functions $\cos x$ and $\sin x$. Then, the integral equation

$$y(x)=\sin x+\cos x+\lambda\int_0^{2\pi}\cos(x+t)\,y(t)\,dt$$

has

1. unique solution for $\lambda=\frac{1}{\pi}$
2. unique solution for $\lambda=-\frac{1}{\pi}$
3. unique solution for $\lambda=\pi$
4. no solution for $\lambda=-\pi$

(GATE 2004)

11. The eigen values λ of the integral equation

$$y(x)=\lambda\int_0^{2\pi}\sin(x+t)\,y(t)\,dt \text{ are}$$

1. $\frac{1}{2\pi},-\frac{1}{2\pi}$
2. $\frac{1}{\pi},-\frac{1}{\pi}$
3. $\pi,-\pi$
4. $2\pi,-2\pi$

(GATE 2005)

12. Solution of the initial value problem

$$\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=F(x),$$

$0\le x\le 1$

$$y(0)=C_0,\ \left(\frac{dy}{dx}\right)_{x=0}=C_1,$$

where, $a_1(x), a_2(x)$ and $F(x)$ are continuous functions on $[0,1]$ may be reduced, in general, to a solution of some linear

1. Fredholm integral equation of first kind
2. Volterra's integral equation of first kind
3. Fredholm integral equation of second kind
4. volterra's integral equation of second kind

(GATE 2006)

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13. Which of the following functions is a solution of the Volterra type integral equation $f(x) = x + \int_0^x [\sin(x-t) f(t)] dt$?

1. $x + \frac{x^3}{3}$ 2. $x - \frac{x^3}{3}$

3. $x + \frac{x^3}{6}$ 4. $x - \frac{x^3}{6}$

(GATE 2006)

14. Which of the following functions is a solution of the Fredholm type equation

$f(x) = x + \int_0^1 [xt\, f(t)] dt$?

1. $\frac{2x}{3}$ 2. $\frac{3x}{2}$

3. $\frac{3x}{4}$ 4. $\frac{4x}{3}$

(GATE 2006)

15. The value of α for which the integral equation $u(x) = \alpha \int_0^1 e^{x-t} u(t) dt$, has a non-trivial solution is

1. -2 2. -1 3. 1 4. 2

(GATE 2007)

16. Suppose $y(x) = \lambda \int_0^{2\pi} y(t) \sin(x+t) dt$, $x \in [0, 2\pi]$ has eigen values $\lambda = \frac{1}{\pi}$ and $\lambda = -\frac{1}{\pi}$ with corresponding eigen functions $y_1(x) = \sin x + \cos x$ and $y_2(x) = \sin x - \cos x$ respectively. Then, the integral equation

$y(x) = f(x) + \frac{1}{\pi} \int_0^{2\pi} y(t) \sin(x+t) dt$,

$x \in [0, 2\pi]$ has a solution, when $f(x)$ is equal to

1. 1 2. $\cos x$

3. $\sin x$ 4. $1 + \sin x + \cos x$

(GATE 2007)

17. The resolvent kernel for the integral equation $u(x) = F(x) + \int_{\log 2}^{x} e^{(t-x)} u(t) dt$, is

1. $\cos(x-t)$ 2. 1

3. e^{t-x} 4. $e^{2(t-x)}$

(GATE 2009)

18. For a continuous function

$y(t) = f(t) + 3\int_0^1 ts\, y(s) ds$ has

1. a unique solution, if $\int_0^1 sf(s) ds \neq 0$

2. no solution, if $\int_0^1 sf(s) ds = 0$

3. infinitely many solutions, if $\int_0^1 sf(s) ds = 0$

4. infinitely many solutions, if $\int_0^1 sf(s) ds \neq 0$

(GATE 2010)

19. The eigen value λ of the following Fredholm integral equation

$y(x)=\lambda\int_0^1 x^2 t\, y(t)dt,$ is

1. -2 2. 2 3. 4 4. -4

(GATE 2011)

20. The resolvent kernel $R(x,t;\lambda)$ for the integral equation $u(x)=x+\lambda\int_0^1 xe^t u(t)dt$ is

1. $\dfrac{xe^t}{1-\lambda}$
2. $\dfrac{\lambda xe^t}{1+\lambda}$
3. $\dfrac{xe^t}{1+\lambda^2}$
4. $\dfrac{xe^t}{1-\lambda^2}$

(GATE 2012)

21. The solution of this integral equation

$u(x)=x+\lambda\int_0^1 xe^t u(t)dt$ is

1. $\dfrac{x+1}{1-\lambda}$
2. $\dfrac{x^2}{1-\lambda^2}$
3. $\dfrac{x}{1+\lambda^2}$
4. $\dfrac{x}{1-\lambda}$

(GATE 2012)

22. For the Volterra type linear integral equation $\phi(x)=x+2\int_0^x e^{x-\zeta}\phi(\zeta)d\zeta$, the resolvent kernel $R(x,\zeta;2)$ of the kernel $e^{x-\zeta}$ is

1. $(x-\zeta)^2 e^{2(x-\zeta)}$
2. $(x-\zeta)e^{x-\zeta}$
3. $e^{3(x-\zeta)}$
4. $e^{(x-\zeta)}$

(CSIR NET June 2011)

23. For the linear integral equation

$\phi(x)=x+\int_0^{\frac{1}{2}}\phi(\xi)d\xi$, the resolvent kernel $R(x,\xi;1)$ is

1. $\dfrac{1}{2}$ 2. 2 3. $\dfrac{3}{2}$ 4. 4

(CSIR NET June 2012)

24. For the homogeneous Fredholm integral equation $\phi(x)=\lambda\int_0^1 e^{x+t}\phi(t)dt$, a non-trivial solution exists, when λ has the value

1. $\lambda=\dfrac{2}{e-1}$
2. $\lambda=\dfrac{1}{e^2+1}$
3. $\lambda=\dfrac{1}{e+1}$
4. $\lambda=\dfrac{2}{e^2-1}$

(CSIR NET Dec 2012)

25. The integral equation

$y(x)=x-\int_1^x xy(t)dt$; $y\in C^1[1,\infty)$ has the solution

1. $y=x(1-\ln x)$
2. $y=xe^{x-\frac{1}{2}}(x-1)+x$
3. $y=xe^{(1-x^2)/2}$
4. $y=x-x\left(e^{x^2}-e\right)$

(CSIR NET June 2013)

26. The integral equation

$\varphi(x)=f(x)+\int_0^1 K(x,y)\varphi(y)dy$

For $K(x,y)=xy^2$ has a solution

1. $\varphi(x)=f(x)$
2. $\varphi(x)=K(x,x)$
3. $\varphi(x)=x^3$

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4. $\varphi(x)=f(x)+\frac{4}{3}x\int_0^1 x^2 f(x)dx$

(CSIR NET Dec 2013)

27. The homogeneous integral equation

$\varphi(x)-\lambda\int_0^1(3x-2)t\varphi(t)dt=0$, has

1. One characteristic number
2. Three characteristic numbers
3. Two characteristic numbers
4. No characteristic number

(CSIR NET June 2014)

28. Let $y:[0,\infty)\to\mathbb{R}$ be twice continuously differentiable and satisfy

$y(x)+\int_0^x(x-s)y(s)ds=\frac{x^3}{6}$. Then

1. $y(x)=\frac{1}{6}\int_0^x s^3\sin(x-s)ds$
2. $y(x)=\frac{1}{6}\int_0^x s^3\cos(x-s)ds$
3. $y(x)=\int_0^x s\sin(x-s)ds$
4. $y(x)=\int_0^x s\cos(x-s)ds$

(CSIR NET Dec 2014)

29. The integral equation

$y(x)=\lambda\int_0^1(3x-2)t\,y(t)dt$, with λ as a parameter, has

1. only one characteristic number
2. two characteristic numbers
3. more than two characteristic numbers
4. no characteristic number

(CSIR NET June 2015)

30. The resolvent kernel $R(x,t,\lambda)$ for the Volterra integral equation

$\varphi(x)=x+\lambda\int_a^x\varphi(s)ds$, is

1. $e^{\lambda(x+t)}$
2. $e^{\lambda(x-t)}$
3. $\lambda e^{(x+t)}$
4. $e^{\lambda xt}$

(CSIR NET Dec 2015)

31. Consider the integral equation

$y(x)=x^3+\int_0^x\sin(x-t)y(t)dt, x\in[0,\pi]$.

Then the value of $y(1)$ is

1. 19/20
2. 1
3. 17/20
4. 21/20

(CSIR NET June 2016)

32. Let ϕ satisfy

$\phi(x)=f(x)+\int_0^x\sin(x-t)\phi(t)dt$. Then ϕ is given by

1. $\phi(x)=f(x)+\int_0^x(x-t)f(t)dt$
2. $\phi(x)=f(x)-\int_0^x(x-t)f(t)dt$
3. $\phi(x)=f(x)-\int_0^x\cos(x-t)f(t)dt$
4. $\phi(x)=f(x)-\int_0^x\sin(x-t)f(t)dt$

(CSIR NET Dec 2016)

33. Let $\phi(x)$ be the solution of

$\int_0^x e^{x-t}\phi(t)\,dt = x,\ x>0$. Then $\phi(1)$ equals

1. -1
2. 0
3. 1
4. 2

(CSIR NET June 2017)

34. Let $u(x,t)$ be a solution of the heat equation $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ in a rectangle $[0,\pi]\times[0,T]$ subject to the boundary conditions $u(0,t)=u(\pi,t)=0,\ 0\le t\le T$ and the initial condition $u(x,0)=\varphi(x),\ 0\le x\le\pi$. If $f(x)=u(x,T)$, then which of the following is true for a suitable kernel $k(x,y)$?

1. $\int_0^\pi k(x,y)\varphi(y)\,dy = f(x),\ 0\le x\le\pi$
2. $\varphi(x)+\int_0^\pi k(x,y)\varphi(y)\,dy = f(x),\ 0\le x\le\pi$
3. $\int_0^x k(x,y)\varphi(y)\,dy = f(x),\ 0\le x\le\pi$
4. $\varphi(x)+\int_0^x k(x,y)\varphi(y)\,dy = f(x),\ 0\le x\le\pi$

(CSIR NET Dec 2017)

35. The resolvent kernel for the integral equation $\phi(x)=x^2+\int_0^x e^{t-x}\varphi(t)\,dt$ is

1. e^{t-x}
2. 1
3. e^{x-t}
4. x^2+e^{x-t}

(CSIR NET June 2018)

------------------------ M C Q ---------------------

1. Given integral equation

$y(x)=\int_0^x (x+t)y(t)\,dt+1$ is

1. A Fredholm integral equation
2. A Volterra's integral equation
3. The required differential equation is $\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-3y=0$
4. The initial equation of required differential equation is $y(0)=1,\ y'(0)=0$

2. Given, $\frac{d^2y}{dx^2}-\sin x\frac{dy}{dx}+e^x y = x$ with the initial conditions $y(0)=1, y'(0)=-1$.

Then,

1. the equivalent integral equation is
$$y(x)=\left[x-\sin x+e^x(x-1)\right]+\int_0^x\left[\sin x-e^x(x-\xi)\right]y(\xi)\,d\xi$$
2. the Volterra's integral equation of first find
3. the Volterra's integral equation of second kind
4. the Fredholm integral equation of first kind

3. Find the solution of

$\phi(x)-\lambda\int_0^x e^{x-y}\phi(y)\,dy = f(x)$

1. $\phi(x)-\lambda\int_0^x e^{x-y}\phi(y)\,dy = f(x)$ is the

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Volterra integral equation of second type

2. $H(x,y;\lambda) = -e^{(1+\lambda)(x-y)}$ is the resolvent kernel for this Volterra integral equation

3. $\phi(x) = f(x) + \lambda\int_0^x e^{(1+\lambda)(x-y)} f(y)\,dy$ is the solution of this Volterra integral equation

4. $J(x,y) = e^x e^{-y}$ is the kernel for Volterra integral equation

4. Given the integral equation

$$\phi(x) = \lambda\int_0^1 \left[\sqrt{(x)}\xi - \sqrt{(\xi)}x\right]\phi(\xi)\,d\xi,$$ then

1. the integral have no real characteristic numbers
2. the integral have real characteristic number
3. it has a trivial solution $\phi(x) = 0$
4. a Fredholm integral equation

5. Given the integral equation

$$\phi(x) = (1+x) + \int_0^x (x-\xi)\phi(\xi)\,d\xi,\ \phi_0(x) = 1,$$

then

1. a Volterra integral equation of first kind
2. a Volterra integral equation of second kind
3. the solution is

$$\phi(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

4. the solution is $\phi(x) = e^x$

6. Given integral equation

$$\phi(x) = \sec^2 x + \lambda\int_0^1 \phi(\xi)\,d\xi$$ is

1. The Volterra integral equation
2. The Fredholm integral equation
3. The solution is $\phi(x) = \sec^2 x + \dfrac{\lambda}{1-\lambda}\tan 1$
4. The solution is $\phi(x) = \sec^2 x + \tan x$

7. Given the integral equation

$$y(x) = \sin x + 2\int_0^x \cos(x-t)\,y(t)\,dt,$$ then

1. $y(x) = xe^x$ is a solution of given integral equation
2. $y(x) = e^x$ is a solution of given integral equation
3. a Volterra integral equation
4. a Fredholm integral equation

8. Given integral equation,

$$y(x) = 1 + \lambda\int_0^1 xt\,y(t)\,dt,$$ then

1. the solution is $y(x) = 1 + \dfrac{3\lambda x}{2(3-\lambda)}, \lambda < 3$
2. the solution is

$$y(x) = 1 + \frac{\lambda x}{2}\left(1 + \frac{\lambda}{3} + \left(\frac{\lambda}{3}\right)^2 + \left(\frac{\lambda}{3}\right)^3 + ... + \infty\right)$$

3. if initial approximation $y_1(x) = 1,$ then

$$y_1(x) = 1 + \frac{\lambda x}{2}$$

4. if initial approximation, $y_1(x) = 1,$ then

$$y_2(x) = 1 + \frac{\lambda x}{2} + \frac{\lambda^2 x}{6}$$

9. Given the integral equation

$\int_0^x e^{x-\xi} y(\xi)\, d\xi = x$, then

1. a Volterra integral equation
2. a Fredholm integral equation
3. $y(x) = 1 - x$ is a solution of integral equation
4. $y(x) = 1 - x$ is not a solution of integral equation

10. Given the integral equation

$\phi(x) = 1 + \int_0^x \phi(\xi)\, d\xi; \phi_0(x) = 0$ is

1. a Volterra integral equation of first kind
2. a Volterra integral equation of second kind
3. $\phi(x) = e^x$ is a solution
4. $\phi(x) = 0$ is a solution

11. Given the equation

$\frac{dy}{dx} = 3\int_0^x \cos 2(x-t)\, y(t)\, dt + 2$, given

$y(0) = 1$ is

1. a Volterra integral equation
2. a Fredholm integral equation
3. the solution is
   $y = 4 + 8x - 3\cos x - 6\sin x$
4. the solution is $y = 4x + 3\sin x + 2\cos x$

12. Given the integral equation

$\int_0^x \frac{y(t)}{\sqrt{x-t}}\, dt = 1 + 2x - x^2$ is

1. the Volterra integral equation
2. the fredholm integral equation
3. the solution is $y = \frac{1}{3\pi\sqrt{x}}\left(3 + 12x - 8x^2\right)$
4. the solution is $y = \frac{1}{3\pi}\left(\sqrt{x} + 2\right)$

13. Given the integral equation

$\int_0^x e^{x-t} y(t)\, dt = e^x + x - 1$, then

1. $y(t) = 2 - t$ is the solution
2. $y(t) = 2 + t$ is the solution
3. Volterra integral equation
4. Fredholm integral equation

14. Given the homogeneous integral equation

$y(x) = \frac{1}{e^2 - 1}\int_0^1 e^{x+t}\, y(t)\, dt$ with degenerate kernel, then

1. has no trivial solution
2. has only trivial solution
3. it does not contains any eigen value or eigen function
4. it contains eigen value of eigen function

15. Given linear integral equation

$\phi(x) = x + \int_0^{\frac{1}{2}} \phi(\xi)\, d\xi$ is

1. Volterra integral equation of first kind
2. Volterra integral equation of second kind
3. $K_3(x,\xi) = \frac{1}{2}\int_0^{\frac{1}{2}} dz = \left(\frac{1}{2}\right)^2$
4. the solution is $\phi(x) = x + \int_0^{\frac{1}{2}} 2\xi\, d\xi = x + \frac{1}{4}$

16. Given the integral equation,

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$\int_0^x \frac{y(\xi)}{\sqrt{(x-\xi)}} d\xi = 1$ is

1. Volterra integral equation of first kind
2. Volterra integral equation of second kind
3. Fredholm integral equation
4. the solution of the equation is $y(x) = \frac{1}{\pi\sqrt{x}}$

17. Given the integral equation

$x = \int_0^x e^{x-\xi}\phi(\xi)d\xi$ is

1. a Volterra equation
2. a Fredholm equation
3. the solution is $\phi(x) = 1 - x$
4. the solution is $\phi(x) = \frac{1}{1-x}$

18. The integral equation, involving a parameter λ,

$\phi(x) = \cos 2x + \lambda\int_0^\pi \cos(x+\zeta)\phi(\xi)d\zeta$

1. a unique solution if $\lambda = 1$, and an infinite number of solution if $\lambda = \frac{2}{\pi}$
2. a unique solution if $\lambda = -1$, and an infinite number of solution if $\lambda = -\frac{2}{\pi}$
3. a unique solution if $\lambda \neq \frac{2}{\pi}$
4. no solution if $\lambda = \pm\frac{2}{\pi}$

(CSIR NET June 2011)

19. For the integral equation

$u(x) = f(x) + \lambda\int_a^b K(x,t)u(t)dt$ to have a continuous solution in the interval $a \le x \le b$, which of the following assumptions are necessary ?

1. $K(x,t) \neq 0$, is real and continuous in the region $a \le x \le b$, $a \le t \le b$ with $|K(x,t)| \le M$
2. $f(x) \neq 0$, is real and continuous in the interval $a \le x \le b$.
3. λ is a constant.
4. $|\lambda| < \frac{1}{M(b-a)}$

(CSIR NET Dec 2011)

20. The integral equation

$y(x) = 1 + \lambda\int_0^{\frac{\pi}{2}} \cos(x-t)y(t)dt$ has

1. a unique solution for $\lambda \neq 4/(\pi+2)$
2. a unique solution for $\lambda \neq 4/(\pi-2)$
3. no solution for $\lambda = 4/(\pi+2)$, but the corresponding homogeneous equation has non-trivial solutions.
4. no solution for $\lambda = 4/(\pi-2)$, but the corresponding homogeneous equation has non-trivial solutions.

(CSIR NET Dec 2011)

21. The initial value problem

$\frac{d^2y}{dx^2} + y = 0; x > 0$, $y(0) = 1$, $y'(0) = 0$.

is equivalent to the Volterra integral equation

1. $y(x)=1+\int_0^x (t-x)y(t)dt$
2. $y(x)=1+\int_0^x (t+x)y(t)dt$
3. $y(x)=1+\int_0^x xt\,y(t)dt$
4. $y(x)=1+\int_0^x (x-t)y(t)dt$

22. The integral equation

$\varphi(x)-\lambda\int_{-1}^{1}\cos[\pi(x-t)]\varphi(t)dt=f(x)$

has

1. a unique solution for $\lambda\neq 1$ when $f(x)=x$
2. no solution for $\lambda\neq 1$ when $f(x)=1$
3. no solution for $\lambda=1$ when $f(x)=x$
4. infinite number of solutions for $\lambda=1$ when $f(x)=1$

(CSIR NET Dec 2012)

23. For the homogeneous Fredholm equation $y(x)=\lambda\int_0^\pi \sin(x+\zeta)y(\zeta)d\zeta$, the eigenvalue λ and the corresponding eigenfunction $y(x)$, involving arbitrary constants *A* and *B*, are

1. $\lambda=\frac{2}{\pi}$, $y(x)=A(\sin x-\cos x)$
2. $\lambda=-\frac{2}{\pi}$, $y(x)=B(\sin x+\cos x)$
3. $\lambda=-\frac{2}{\pi}$, $y(x)=B(\sin x-\cos x)$
4. $\lambda=\frac{2}{\pi}$, $y(x)=A(\sin x+\cos x)$

(CSIR NET June 2013)

24. Let λ_1,λ_2 be the characteristic numbers and f_1, f_2 the corresponding eigen functions for the homogeneous integral equation

$\varphi(x)-\lambda\int_0^1 (xt+2x^2)\varphi(t)dt=0$. Then

1. $\lambda_1=-18-6\sqrt{10},\ \lambda_2=-18+6\sqrt{10}$
2. $\lambda_1=-36-12\sqrt{10},\ \lambda_2=-36+12\sqrt{10}$
3. $\int_0^1 f_1(x)\,f_2(x)dx=1$
4. $\int_0^1 f_1(x)\,f_2(x)dx=0$

(CSIR NET June 2014)

25. If $y:[0,\infty)\to[0,\infty)$ is a continuously differentiable function satisfying

$y(t)=y_0-\int_0^t y(s)ds$ for $t\geq 0,$ then

1. $y^2(t)=y^2(0)+\left(\int_0^t y(s)ds\right)^2-2y(0)\int_0^t y(s)ds$
2. $y^2(t)=y^2(0)+2\int_0^t y^2(s)ds$
3. $y^2(t)=y^2(0)-\int_0^t y(s)ds$
4. $y^2(t)=y^2(0)-2\int_0^t y^2(s)ds$

(CSIR NET June 2014)

26. Let $u\in c^2([0,1])$ satisfy for some $\lambda\neq 0$ and $a\neq 0$

$u(x)+\frac{\lambda}{2}\int_0^1 |x-s|u(s)ds=ax+b$. Then *u* also satisfies

1. $\frac{d^2u}{dx^2}+\lambda u=0$
2. $\frac{d^2u}{dx^2}-\lambda u=0$
3. $\frac{du}{dx}-\frac{\lambda}{2}\int_0^1 \frac{x-s}{|x-s|}u(s)ds=a$

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4. $\frac{du}{dx}+\frac{\lambda}{2}\int_0^1 \frac{x-s}{|x-s|}u(s)ds=a$

(CSIR NET Dec 2014)

27. For the integral equation

$y(x)=1+x^3+\int_0^x K(x,t)y(t)dt$ with

kernel $K(x,t)=2^{x-t}$, the iterated kernel

$K_3(x,t)$ is

1. $2^{x-t}(x-t)^2$
2. $2^{x-t}(x-t)^3$
3. $2^{x-t-1}(x-t)^2$
4. $2^{x-t-1}(x-t)^3$

(CSIR NET June 2015)

28. Let $y:[0,\infty)\to[0,\infty)$ be a continuously differentiable function satisfying

$y(t)=y(0)+\int_0^t y(s)ds$ for $t\ge 0$. Then

1. $y^2(t)=y^2(0)+\int_0^t y^2(s)ds$
2. $y^2(t)=y^2(0)+2\int_0^t y^2(s)ds$
3. $y^2(t)=y^2(0)+\int_0^t y(s)ds$
4. $y^2(t)=y^2(0)+\left(\int_0^t y(s)ds\right)^2+2y(0)\int_0^t y(s)ds$

(CSIR NET Dec 2015)

29. Let λ_1,λ_2 be the characteristic numbers and f_1, f_2 be the corresponding eigenfunctions for the homogeneous integral equation

$\varphi(x)-\lambda\int_0^1(2xt+4x^2)\varphi(t)dt=0$. Then

1. $\lambda_1\neq\lambda_2$
2. $\lambda_1=\lambda_2$
3. $\int_0^1 f_1(x)f_2(x)dx=0$
4. $\int_0^1 f_1(x)f_2(x)dx=1$

(CSIR NET Dec 2015)

30. The curve $y=y(x)$, passing through the point $(\sqrt{3},1)$ and defined by the following property (Voltera integral equation of the first kind) $\int_0^y \frac{f(v)dv}{\sqrt{y-v}}=4\sqrt{y}$, where

$f(y)=\sqrt{1+\frac{1}{y'^2}}$, is the part of a

1. straight line
2. circle
3. parabola
4. cycloid

(CSIR NET June 2016)

31. The integral equation

$\phi(x)-\frac{2}{\pi}\int_0^{\pi}\cos(x+t)\phi(t)\,dt=f(x)$ has

infinitely many solutions if

1. $f(x)=\cos x$
2. $f(x)=\cos 3x$
3. $f(x)=\sin x$
4. $f(x)=\sin 3x$

(CSIR NET Dec 2016)

32. Which of the following are the characteristic numbers and the corresponding eigenfunctions for the Fredholm homogeneous equation whose

kernel is $K(x,t)=\begin{cases}(x+1)t, & 0\le x\le t\\ (t+1)x, & t\le x\le 1\end{cases}$?.

1. $1, e^x$
2. $-\pi^2, \pi\sin\pi x+\cos\pi x$
3. $-4\pi^2, \pi\sin\pi x+\pi\cos 2\pi x$
4. $-\pi^2, \pi\cos\pi x+\sin\pi x$

(CSIR NET Dec 2016)

33. Let $y(x)$ be the solution of the integral

equation $y(x)=x-\int_0^x xt^2 y(t)\,dt,\ x>0$.

Then the value of the function $y(x)$ at

$x=\sqrt{2}$ is equal to

1. $\frac{1}{\sqrt{2e}}$
2. $\frac{e}{2}$
3. $\frac{\sqrt{2}}{e^2}$
4. $\frac{\sqrt{2}}{e}$

(CSIR NET June 2017)

34. The solutions for $\lambda=-1$ and $\lambda=3$ of the integral equation

$y(x)=1+\lambda\int_0^1 K(x,t)y(t)\,dt,$ where

$K(x,t)=\begin{cases}\cosh x\sinh t, & 0\le x\le t\\ \cosh t\sinh x, & t\le x\le t\end{cases}$

are, respectively

1. $-\frac{x^2}{2}+\frac{3}{2}-\tanh 1$ and $\frac{1}{4}\left(\frac{3\cos 2x}{\cos 2-2\sin 2\tanh 1}+1\right)$
2. $-\frac{x^2}{2}+\frac{3}{2}-\tanh 1$ and $\frac{1}{4}\left(\frac{3\cosh 2x}{\cosh 2-2\sinh 2\tanh 1}+1\right)$
3. $-\frac{x^2}{2}+\frac{3}{2}-\tanh 1$ and $\frac{1}{4}\left(\frac{3\cosh 2x}{\cosh 2-2\sinh 2\tanh 1}-1\right)$
4. $\frac{x^2}{2}+\frac{3}{2}-\tanh 1$ and $\frac{1}{4}\left(\frac{3\cos 2x}{\cos 2-2\sin 2\tanh 1}-1\right)$

(CSIR NET June 2017)

35. Let φ be the solution of the integral

equation $\frac{1}{2}\varphi(x)-\int_0^1 e^{x-y}\varphi(y)\,dy=x^2$

$0\le x\le 1$. Then

1. $\varphi(0)=20e^{-1}-8$
2. $\varphi(0)=20e-8$
3. $\varphi(1)=22-8e$
4. $\varphi(1)=22-8e^{-1}$

(CSIR NET Dec 2017)

36. Consider a non-zero, real-valued polynomial function $p(x)=a_0+a_1x+a_2x^2$ of degree at most 2. Let $y=y(x)$ be a solution of the integral equation

$y=p(x)+\int_0^x y(t)\sin(x-t)\,dt$

Which of the following statements are necessarily correct ?

1. $y(x)$ is a polynomial function of degree ≤ 2
2. $y(x)$ is a polynomial function of degree ≤ 4
3. If $a_1\ne 0$ and $a_0+2a_2=0$, then

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$y'(0)=0$

4. If $a_1 \neq 0$ and $a_0+2a_2=0$, then $y''(0)=0$

(CSIR NET Dec 2017)

37. The values of λ for which the following equation has a non-trivial solution

$$\phi(x)=\lambda\int_0^{\pi} K(x,t)\phi(t)\,dt,\ \ 0\le x\le \pi$$

where $K(x,t)=\begin{cases}\sin x\cos t, & 0\le x\le t\\ \cos x\sin t, & t\le x\le \pi\end{cases}$ are

1. $\left(n+\frac{1}{2}\right)^2-1,\ n\in\mathbb{N}$
2. $n^2-1,\ n\in\mathbb{N}$
3. $\frac{1}{2}(n+1)^2-1,\ n\in\mathbb{N}$
4. $\frac{1}{2}(2n+1)^2-1,\ n\in\mathbb{N}$

(CSIR NET June 2018)

38. Consider the integral equation

$$\phi(x)=\lambda\int_0^{\pi}[\cos x\cos t-2\sin x\sin t]\phi(t)\,dt$$

$$+\cos 7x,\quad 0\le x\le \pi$$

Which of the following statements are true ?

1. For every $\lambda\in\mathbb{R}$, a solution exists
2. There exists $\lambda\in\mathbb{R}$ such that solution does not exist
3. There exists $\lambda\in\mathbb{R}$ such that there are more than one but finitely many solutions
4. There exists $\lambda\in\mathbb{R}$ such that there are infinitely many solutions

(CSIR NET June 2018)

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Assignment
SCQ

1. C
2. 4
3. 1
4. 3
5. 1
6. 1
7. 1
8. 4
9. 3
10. 3
11. 2
12. 4
13. 3
14. 2
15. 3
16. 1
17. 2
18. 3
19. 3
20. 1
21. 4
22. 3
23. 2
24. 4
25. 3
26. 4
27. 4
28. 3
29. 4
30. 2
31. 4
32. 1
33. 1
34. --
35. 2

MCQ

1. 2,3,4
2. 1,3
3. 1,3,4
4. 1,3,4
5. 2,3,4
6. 2,3
7. 1,3
8. 1,2,3,4
9. 1,3
10. 2,3
11. 1,3
12. 1,3
13. 1,3
14. 2,3
15. 3,4
16. 1,4
17. 1,3
18. 1
19. 1,2,3,4
20. 3
21. 1
22. 1,3,4
23. 3,4
24. 1,4
25. 1,4
26. 1,4
27. 3
28. 2,4
29. 1,3
30. 1
31. 2,3,4
32. 1,4
33. 4
34. 2
35. 1,3
36. 2,4
37. 1
38. 1,2

1. Simple variational problems

Variational Calculus is the branch of mathematics concerned with the problem of finding a function for which the value of a certain integral is either the largest or the smallest possible. That integral is technically known as a functional.

Variational Calculus deals with functionals. Briefly, a functional is a function of a function. The difference between a functional and an ordinary function can be appreciated in the table.

1.(a) Input : argument x (independent variable); Function operator; Output : function value y (dependent variable) — **Functions**

$$\boxed{x} \longrightarrow \boxed{f} \longrightarrow \boxed{y = y(x) = f(x)}$$

1.(b) Input 1 : argument x (independent variable); Input 2 : function $y = y(x)$ (primary dependent variable); Functional operator; Output : functional value J (a scalar)

$$\boxed{x} \longrightarrow \boxed{f} \longrightarrow \boxed{y = f(x)} \longrightarrow \boxed{J} \longrightarrow \boxed{J[y] = J(x, y)}$$

1.(c) Input 1 : argument x (independent variable); Input 2 : function $y = y(x)$ (primary dependent variable); Functional operator; Output : functional value J (a scalar)

$$\boxed{x} \longrightarrow \boxed{f} \longrightarrow \boxed{y = f(x)} \longrightarrow \boxed{J} \longrightarrow \boxed{J[y] = J(x, y, y')}$$

Input 3 : derivative of primary dependent value $\boxed{y' = dy/dx}$ — **Functionals**

We shall consider the functional

$$I\left(y(x)\right) = \int_a^b F(x, y, y')\,dx \quad ; \quad y(a) = y_1,\ y(b) = y_2$$

where x is independent variable, y is dependent variable and y' is first derivative of the dependent variable.

Three forms of Euler's equation :

(i) $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$

(ii) $\dfrac{d}{dx}\left(F - y'\dfrac{\partial F}{\partial y'}\right) - \dfrac{\partial F}{\partial x} = 0$

(iii) $\dfrac{\partial F}{\partial y} - \dfrac{\partial^2 F}{\partial x\, \partial y'} - y'\dfrac{\partial^2 F}{\partial y\, \partial y'} - y''\dfrac{\partial^2 F}{\partial y'^2} = 0$

Note : For a functional extremal can be unique, infinite or does not exist depending on the boundary conditions.

Def. $C^n[a,b]$ is the set of all functions having continuous n^{th} order derivative in $[a,b]$

Some results from calculus :

1. The distance between two points (x_1, y_1) and (x_2, y_2) along a curve $y = y(x)$ is given by
$I[y(x)] = \int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx$.
2. When a curve joining $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is revolved about *x*-axis, the area of the surface of revolution is given by $S[y(x)] = 2\pi \int_{x_1}^{x_2} y\sqrt{1+y'^2}\,dx$.
3. Time taken by a particle in sliding from (x_1, y_1) to (x_2, y_2) along a curve $y = y(x)$ is given by
$T[y(x)] = \frac{1}{\sqrt{2g}} \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{\sqrt{y}}\,dx$ where g is gravitational constant.
4. Cycloid is formed during rolling of any circle. Let *a* be the radius of that rolling circle then parametric equation of cycloid is $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$

Type : Integrand contains y only

Example : Find the extremal of the functional $I\left(y(x)\right) = \int_0^1 \left(y - \frac{y^3}{3}\right)dx$ subject to the conditions $y(0) = 1$ and $y(1) = 1$.

Solution : $F(x, y, y') = y - \frac{y^3}{3}$, $y(0) = 1$, $y(1) = 1$

Euler's equation : $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$\Rightarrow \quad 1 - y^2 - \frac{d}{dx}(0) = 0 \quad \Rightarrow \quad y^2 = 1 \quad \Rightarrow \quad y(x) = 1$

$y(0) = 1$, $\quad y(1) = 1$, both the boundary conditions are satisfied.

Hence the extremal is $y(x) = 1$.

Exercise 1.1

1. Find the extremal of the functional $I\left(y(x)\right) = \int_0^1 \left(y - \frac{y^3}{3}\right)dx$ subject to the conditions $y(0) = 0$ and $y(1) = 1$.
2. Find the extremal of the functional $I\left(y(x)\right) = \int_a^b \left(y - \frac{y^3}{3}\right)dx$ subject to the conditions $y(a) = y_1$

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and $y(b) = y_2$. What are the conditions on y_1 and y_2 so that given functional can have an extremum ?

3. Find the extremal of the functional $I(y(x)) = \int_a^b \left(y + \frac{y^3}{3} \right) dx$.

4. Find the extremal of the functional $I(y(x)) = \int_1^2 y'^2 dx$ subject to the boundary conditions $y(1) = 0$ and $y(2) = 0$. Also tell that extremal curve is a minimum curve or maximum curve.

5. Find the extremal of the functional $I(y(x)) = \int_0^1 y'^3 dx$ subject to the boundary conditions $y(0) = 0$ and $y(1) = 1$.

6. Find the extremal of the functional $I[y(x)] = \int_0^1 \left(2e^y - y^2 \right) dx$ subject to the conditions $y(0) = 1$ and $y(1) = e$.

Answers

1. No extremal	2. $y_1 = y_2 = 1$ *or* $y_1 = y_2 = -1$	3. No extremal
4. $y(x) = 0$, minimum curve	5. No extremal	6. No extremal

Type : Integrand contains x and y only

Example : Find the extremal of the functional $I[y(x)] = \int_1^3 (3x - y)\, y\, dx$ subject to the conditions $y(1) = \frac{3}{2}$, $y(3) = \frac{9}{2}$.

Solution : $F(x, y, y') = (3x - y) y$, $y(1) = \frac{3}{2}$, $y(3) = \frac{9}{2}$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) = 0 \quad \Rightarrow \quad 3x - 2y - 0 = 0 \quad \Rightarrow \quad y(x) = \frac{3}{2}x$$

$y(1) = \frac{3}{2}$ and $y(3) = \frac{9}{2}$, both the boundary condtions are satisfied.

Hence the extremal is $y(x) = \frac{3x}{2}$.

Exercise 1.2

1. Find the extremal of the functional $I[y(x)] = \int_1^3 (3x - y)\, y\, dx$ subject to the conditions $y(1) = 1$, $y(3) = \frac{9}{2}$.

2. Find the extremal of the functional $I[y(x)] = \int_0^{\frac{\pi}{2}} y(2x - y)dx$; $y(0) = 0$, $y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

3. Find the extremal of the functional $I[y(x)] = \int_0^1 (5x + 2y)\, y\, dx$ subject to the conditions $y(0) = 0$, $y(1) = \frac{5}{4}$.

4. What changes in the values of $y(0)$ and $y(1)$ should be made in above problem so that it may have an extremal ?

Answers

1. No extremal 2. $y = x$ 3. No extremal 4. $y(1) = -\frac{5}{4}$ and $y(0)$ is unchanged.

Type : Integrand is of the form $M(x, y) + N(x, y)y'$

Example : Find the extremals of the functional $\int_0^1 (e^y + xy')dx$ that satisfy the boundary conditions :

$y(0) = 0$, $y(1) = 0$

Solution : $F(x, y, y') = e^y + xy'$, $y(0) = 0$, $y(1) = 0$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \Rightarrow \quad e^y - \frac{d}{dx}(x) = 0$$

$$\Rightarrow \quad e^y - 1 = 0 \quad \Rightarrow \quad e^y = 1$$

Taking log both side, we get $y(x) = 0$

$y(0) = 0 = y(1)$, both the boundary conditions are satisfied.

Hence the extremal is $y(x) = 0$

Exercise 1.3

1. Find the extremals of the functional $\int_0^1 \left(xy + y^2 - 2y^2 y'\right)dx$, $y(0) = 1$, $y(1) = 2$

2. Find the extremals of the functional $I[y(x)] = \int_0^1 (y^2 + x^2 y')dx$; $y(0) = 0$, $y(1) = a$

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3. Test on extremum the functional $I[y(x)] = \int_{x_1}^{x_2} (y + xy')dx$ subject to boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$

4. Test on extremum the functional $I[y(x)] = \int_a^b (y + y')dx$.

5. Test on extremum the functional $I[y(x)] = \int_0^1 x\,y\,y'\,dx$ subject to boundary conditions $y(0) = 0$ and $y(1) = 1$.

Answers

1. No extremal
2. If $a = 1$ then the extremal is $y = x$, and if $a \neq 1$ then no extremal.
3. Problem is meaningless and so no extremal.
4. No extremal
5. No extremal

Type : Integrand contains y' only

Example : Find the extremals of $\int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx$ subject to the conditions $y(x_1) = y_1$ and $y(x_2) = y_2$.

Solution : $F(x, y, y') = \sqrt{1+y'^2}$, $y(x_1) = y_1$, $y(x_2) = y_2$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \Rightarrow \quad 0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\Rightarrow \quad \frac{y'}{\sqrt{1+y'^2}} = c^2 \quad \Rightarrow \quad y'^2 = c\left(1+y'^2\right)$$

$$\Rightarrow \quad y'^2(1-c) = c \quad \Rightarrow \quad y'^2 = \frac{c}{1-c} = c \quad \Rightarrow \quad y' = \sqrt{c} \quad \Rightarrow \quad y' = c_1 \quad \text{(say)}$$

$$\Rightarrow \quad y(x) = c_1 x + c_2$$

$$y(x_1) = y_1 \quad \Rightarrow \quad y_1 = c_1 x_1 + c_2 \quad \text{.......(1)}$$

$$y(x_2) = y_2 \quad \Rightarrow \quad y_2 = c_1 x_2 + c_2 \quad \text{.......(2)}$$

By equation (1) and (2), we get $c_1 = \dfrac{y_1 - y_2}{x_1 - x_2}$, $c_2 = y_1 - x_1\left(\dfrac{y_1 - y_2}{x_1 - x_2}\right)$

$$\therefore \quad y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Exercise 1.4

1. Find the extremals of $\int_{x_1}^{x_2} \sqrt{1+|y'(x)|^2}\,dx$ subject to the conditions $y(x_1)=y_1$ and $y(x_2)=y_2$.
2. By using Euler's equation, find the curves passing through (0,3) and (4,11) such that its length between the given points is shortest.
3. Find the extremal of $\int_0^1 y'dx$ subject to the conditions $y(0)=0$, $y(1)=1$.
4. Find the extremal of $\int_0^1 y'^2\,dx$ subject to the conditions $y(0)=0$, $y(1)=1$.

Answers

1. $y-y_1=\dfrac{y_2-y_1}{x_2-x_1}(x-x_1)$ 2. $y=2x+3$
3. Problem is meaningless and so no extremal 4. $y=x$

Type : Integrand contains x and y' only

Example : Find the extremal of the functional $I[y(x)]=\int_{1/2}^{1} x^2y'^2\,dx$; $y\left(\frac{1}{2}\right)=1, y(1)=2$

Solution : $F(x,y,y')=x^2y'^2$, $y\left(\frac{1}{2}\right)=1,\ y(1)=2$

$$0-\frac{d}{dx}\left(2x^2y'\right)=0$$

$\Rightarrow \quad x^2y'=c \quad \Rightarrow \quad x^2\dfrac{dy}{dx}=c \quad \Rightarrow \quad dy=\dfrac{c}{x^2}dx$

$\Rightarrow \quad y=c_1\dfrac{x^{-1}}{-1}+c_2 \quad \Rightarrow \quad y=\dfrac{-c_1}{x}+c_2$

$y\left(\frac{1}{2}\right)=1 \quad \Rightarrow \quad 1=-2c_1+c_2$ …..(1)

$y(1)=2 \quad \Rightarrow \quad 2=-c_1+c_2$ …..(2)

By equation (1) and (2), we get $c_1=1$ and $c_2=3$

$\therefore \quad y(x)=\dfrac{-1}{x}+3 \quad \Rightarrow \quad y'(x)=\dfrac{1}{x^2}$

and $I(y(x))=\int_{\frac{1}{2}}^{1} x^2-\dfrac{1}{x^4}dx=\int_{\frac{1}{2}}^{1}\dfrac{1}{x^2}dx=-\left|x^{-1}\right|_{\frac{1}{2}}^{1}=-[1-2]=1$

Value $=1$

Exercise 1.5

1. Find the extremal of the functional $I[y(x)] = \int_{x_1}^{x_2} \left(xy' + y'^2 \right) dx$.

2. Find the extremals and extremum value of the functional $\int_0^2 \left(x - y' \right)^2 dx$; $y(0) = 0$ and $y(2) = 4$

3. Find the extremal of the functional $I[y(x)] = \int_0^4 \left(xy' - y'^2 \right) dx$; $y(0) = 0$, $y(4) = 3$

4. Find the extremal of the functional $I[y(x)] = \int_{x_1}^{x_2} y' \left(1 + x^2 y' \right) dx$

5. From among the curves connecting the points $P_1(1,3)$ and $P_2(2,5)$, find the curve on which an extremum of the functional $\int_1^2 y' \left(1 + x^2 y' \right) dx$ can be obtained.

6. Find the extremal of the functional $\int_0^1 \left(x + y'^2 \right) dx$ that satisfy the boundary conditions $y(0) = 1$, $y(1) = 2$.

7. Find the extremal of the functional $I[y(x)] = \int_{x_1}^{x_2} x^n \left(\frac{dy}{dx} \right)^2 dx$ where $n \neq 1$, passing through the fixed points (x_1, y_1) and (x_2, y_2).

8. Show that the extremal of the functional $I[y(x)] = \int_0^2 \frac{y'^2}{x} dx$, $y(0) = 0$ and $y(2) = 1$ is a parabola.

9. Find the extremal of the functional $\int_{x_1}^{x_2} \frac{y'^2}{x^3} dx$

10. Find the extremal of the functional $\int_1^2 \frac{x^3}{y'^2} dx$ with $y(1) = 0$ and $y(2) = 3$

11. Find the extremal of the functional $\int_0^4 \frac{(y')^{-2}}{x^{-3}} dx$; $y(0) = 0$, $y(4) = 4$

12. Find the extremal of the functional $\int_1^2 \frac{\dot{x}^2}{t^3} dt$ with $x(1) = 3$ and $x(2) = 18$

13. Find the extremal of the functional $I[y(x)] = \int_{x_1}^{x_2} \frac{\sqrt{1 + y'^2}}{x} dx$

14. Show that the curve through (1,0) and (2,1) which minimizes $\int_1^2 \frac{\sqrt{1+y'^2}}{x}dx$ is a circle.

15. Find the extremal of the functional $I[y(x)]=\int_a^b x\sqrt{1+y'^2}dx$

16. Find the extremal of the functional $I[y(x)]=\int_a^b \sqrt{x}\sqrt{1+y'^2}dx$

Answers

1. $y=c_1x+c_2-\frac{x^2}{4}$ 2. $y=x+\frac{x^2}{2}$; value = 2 3. $y=\frac{x^2-x}{4}$

4. $y=\frac{c_1}{x}+c_2$ 5. $y=7-\frac{4}{x}$ 6. $y=x+1.$

7. $y=\frac{y_1-y_2}{x_1^{1-n}-x_2^{1-n}}x^{1-n}+\frac{y_2x_1^{1-n}-y_1x_2^{1-n}}{x_1^{1-n}-x_2^{1-n}}$ 8. $x^2=4y$ 9. $y=c_1x^4+c_2$

10. $y=x^2-1$ 11. $y=\frac{x^2}{4}$ 12. $x=t^4+2$

13. $x^2+\left(y-c_1\right)^2=c_2{}^2$ 14. $x^2+(y-2)^2=5$ 15. $x=c_1\cosh\left(\frac{y+c_2}{c_1}\right)$

16. $y=2\sqrt{c_1}\sqrt{x-c_1}+c_2$

Type : Integrand contains y and y' only

Example : Find the extremal of the functional $I\left(y\left(x\right)\right)=\int_0^1\left(y^2+y'^2\right)dx,$ where $y\left(0\right)=0,\ y\left(1\right)=1$

Solution : $F\left(x,y,y'\right)=y^2+y'^2$, $y(0)=0,\ \ y(1)=1$

$2y-\frac{d}{dx}\left(2y'\right)=0 \quad\Rightarrow\quad y-y''=0$

$D^2-1=0 \quad\Rightarrow\quad D=\pm1$

$\Rightarrow \quad y(x)=c_1e^x+c_2e^{-x}$

$y\left(0\right)=0 \quad\Rightarrow\quad c_1+c_2=0 \quad\Rightarrow\quad c_1=-c_2$ ……(1)

$y\left(1\right)=1 \quad\Rightarrow\quad c_1e+c_2e^{-1}=1$ ……(2)

By equation (1) and (2), we get $c_1\left(e-e^{-1}\right)=1.$

Now divide both side by 2, we get

$c_1\left(\frac{e-e^{-1}}{2}\right)=\frac{1}{2} \quad\Rightarrow\quad c_1\sinh1=\frac{1}{2} \quad\Rightarrow\quad c_1=\frac{1}{2\sinh1},\ c_2=-\frac{1}{2\sinh1}$

$\Rightarrow \quad y(x)=\frac{1}{2\sinh1}\left(e^x-e^{-x}\right)=\frac{\sinh x}{\sinh1}$

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Exercise 1.6

1. Obtain the Euler's equation for the extremal of the functional $I(y(x)) = \int_{x_1}^{x_2} \left(ay^2 + by'^2 + cyy'\right) dx$

2. Find the extremal of the functional $I(y(x)) = \int_{x_1}^{x_2} \left(y^2 - yy' + y'^2\right) dx$

3. Find the extremal of the functional $I(y(x)) = \int_{0}^{\frac{\pi}{2}} \left(\dot{x}^2 - x^2\right) dt$, $x(0) = 0$, $x\left(\frac{\pi}{2}\right) = 1$

4. Find the extremal of the functional $I[y(x)] = \int_{x_1}^{x_2} \left(y'^2 + 12yy' - 16y^2\right) dx$

5. Find the extremal of the functional $I[y(x)] = \int_{0}^{1} \left(y'^2 + 4y^2\right) dx$, $y(0) = e^2$, $y(1) = 1$

6. Find the extremal of the functional $J\left[y(x)\right] = \int_0^{\frac{\pi}{4}} \left(y'^2 - y^2\right) dx$ that satisfy the boundary conditions

   $y(0) = 1,\ y\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.$

7. Find the extremals and the stationary function of the functional $J\left[y(x)\right] = \int_0^{\pi} \left(y'^2 - y^2\right) dx$ that satisfy the boundary conditions $y(0) = 1, y(\pi) = -1.$

8. Find the extremals and the stationary function of the functional $J\left[y(x)\right] = \int_0^{2\pi} \left(y'^2 - y^2\right) dx$ that satisfy the boundary conditions $y(0) = 1,\ y(2\pi) = 1.$

9. Find the extremal of the functional $I[y(x)] = \int_{0}^{1} yy'^2\, dx$; $y(0) = 1$, $y(1) = 4^{\frac{1}{3}}$

10. Find the extremal of the functional $I[y(x)] = \int_{0}^{1} \frac{1 + y^2}{y'} dx$ passing through the origin and the point (1,1).

11. Find the extremal of the functional $I[y(x)] = \int_{x_1}^{x_2} \frac{1 + y^2}{y'^2} dx$

12. Find the extremal of the functional $\int_{x_1}^{x_2} y\sqrt{1 + y'^2}\ dx$

13. Find the extremal of the functional $I[y(x)] = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{y}\, dx$

14. Find the extremal of the functional $I[y(x)] = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{\sqrt{y}}\, dx$

Answers

1. $y'' - \frac{a}{b} y = 0$ 2. $y = c_1 e^{x} + c_2 e^{-x}$ 3. $x = \sin t$

4. $y = c_1 \sin(4x + c_2)$ or $y = c_1 \cos(4x + c_2)$ or $y = c_1 \cos 4x + c_2 \sin 4x$ 5. $y = e^{2(1-x)}$

6. $y = \cos x$ 7. $y = \cos x + c \sin x.$ 8. $y = \cos x + c \sin x.$

9. $y = (x+1)^{\frac{2}{3}}$ 10. $y = \tan\left(\frac{\pi x}{4}\right)$ 11. $y = \sin h(c_1 x + c_2)$

12. $y = c_1 \cosh\left(\frac{x + c_2}{c_1}\right)$ 13. $(x - c_1)^2 + y^2 = c_2^2$ 14. $x = a(\theta - \sin\theta), \mathrm{y} = \mathrm{a}(1 - \cos\theta)$

Type : Integrand is of the form $\boldsymbol{y^2 + y'^2 + f(x)y}$ or $\boldsymbol{y^2 - y'^2 - f(x)y}$ or $\boldsymbol{y'^2 - y^2 + f(x)y}$

Example : Find the extremals of the functional $I(y(x)) = \int_0^{\pi} \left(y'^2 - y^2 + 4y\cos x\right) dx$ with

$y(0) = 0, \quad y(\pi) = 0.$

Solution : $F(x, y, y') = y'^2 - y^2 + 4y\cos x, \qquad y(0) = y(\pi) = 0$

$-2y + 4\cos x - \frac{d}{dx}(2y') = 0$

$\Rightarrow \qquad -y + 2\cos x - y'' = 0 \qquad \Rightarrow \qquad y'' + y = 2\cos x$

A.E. $\quad D^2 + 1 = 0 \qquad \Rightarrow \qquad D = \pm i$

$\Rightarrow \qquad \text{C.F.} = c_1 \cos x + c_2 \sin x$

P.I. $\quad = \frac{1}{D^2 + 1}(2\cos x)$, (case of failure)

$= \frac{x}{2D}(2\cos x) = x \sin x$

$y(x) = c_1 \cos x + c_2 \sin x + x \sin x\,, \qquad y(0) = 0 \quad \Rightarrow \quad c_1 = 0$

$\Rightarrow \qquad y(x) = c_2 \sin x + x \sin x\,, \qquad y(\pi) = 0 \quad \Rightarrow \quad c_2(0) = 0$

$\Rightarrow \qquad c_2$ is arbitrary.

$\Rightarrow \qquad y(x) = (c + x)\sin x$

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Exercise 1.7

1. Find the extremals of the functional $I\left(y(x)\right)=\int_0^{\pi}\left(y^2-y'^2-8y\cos h\,x\right)dx$ with $y(0)=2$, $y\left(\frac{\pi}{2}\right)=2\cosh\frac{\pi}{2}$.

2. Find the extremals of the functional $I\left(y(x)\right)=\int_0^{\pi}\left(y'^2-y^2+2xy\right)dx$ with $y(0)=0, y\left(\frac{\pi}{2}\right)=0$.

3. Find the extremals of the functional $I\left(y(x)\right)=\int_0^{\pi}\left(y^2-y'^2-2y\sin x\right)dx$ with $y(0)=0, y\left(\frac{\pi}{2}\right)=1$.

4. Find the extremals of the functional $I\left(y(x)\right)=\int_0^{\pi}\left(y'^2-y^2+4y\sin^2 x\right)dx$ with $y(0)=y\left(\frac{\pi}{2}\right)=\frac{1}{3}$.

5. Find the extremals of the functional $I\left(y(x)\right)=\int_0^{\pi}\left(y^2+y'^2+2ye^x\right)dx$.

6. Find the extremals of the functional $I\left(y(x)\right)=\int_0^{\pi}\left(y^2+y'^2+2y\sec h\,x\right)dx$.

Answers

1. $y(x)=2\cos h\,x$
2. $y(x)=x-\frac{\pi}{2}\sin x$
3. $y(x)=\sin x-\frac{x}{2}\cos x$
4. $y(x)=-\cos x-\frac{1}{3}\sin x+\frac{1}{3}\cos 2x+1$
5. $y(x)=c_1e^x+c_2e^{-x}+\frac{x}{2}e^x$
6. $y(x)=c_1\cosh x+c_2\sinh x+x\sinh x-\cosh x\log 2\cosh x$

Integrand contains all x, y, y' and not of any of the above forms

Example : Find the extremal of the functional $I[y(x)]=\int_0^{e}\left(xy'^2+yy'\right)dx$ subject to the conditions $y(1)=0$, $y(e)=1$.

Solution : $F\left(x,y,y'\right)=xy'^2+yy'$, $\quad y(1)=0,\quad y(e)=1$

$$y'-\frac{d}{dx}\left(2xy'+y\right)=0$$

$\Rightarrow \quad y'-2xy''-2y'-y'=0 \qquad \Rightarrow \quad xy''+y'=0$

$\Rightarrow \quad x^2y''+xy'=0$

Put $x = e^z \Rightarrow \quad z = \log x$

$D^2 - D + D = 0 \quad \Rightarrow \quad D^2 = 0 \quad \Rightarrow \quad D = 0,0$

$\therefore \quad y(z) = c_1 + c_2 z$

$y(x) = c_1 + c_2 \log x$, $\quad y(0) = 0 \quad \Rightarrow \quad c_1 = 0$

$\therefore \quad y(x) = c_2 \log x$, $\quad y(e) = 1 \quad \Rightarrow \quad 1 = c_2 \quad \Rightarrow y(x) = \log x$

Exercise 1.8

1. Find the extremal of the functional $I[y(x)] = \int_0^1 \left(y'^2 + 12xy \right) dx$ with $y(0) = 0$, $y(1) = 1$.
2. On what curves can be functional $J\left[y(x) \right] = \int_1^2 \left(y'^2 - 2xy \right) dx; y(1) = 0, y(2) = -1$ attain an extremum ?
3. Find the stationary function of the functional $I\left[y(x) \right] = \int_{-1}^0 \left(12xy - y'^2 \right) dx$ which is determined by the boundary conditions $y(-1) = 1, y(0) = 0$.
4. Find the extremal of the functional $J\left[y(x) \right] = \int_{-1}^1 \left(y'^2 - 2xy \right) dx$ that satisfy the conditions $y(-1) = -1, y(1) = 1$.
5. Show that the Euler's equation for the functional $\int_{x_1}^{x_2} \left(a(x) y'^2 + 2b(x) yy' + c(x) y^2 \right) dx$ is a second order linear differential equation.

Answers

1. $y = x^3$
2. $y = \dfrac{x}{6}\left(1 - x^2\right)$.
3. $y = -x^3$
4. $y = \dfrac{7}{6} x - \dfrac{x^3}{6}$
5. $a(x) y'' + a'(x) y' + \left(b'(x) - c(x) \right) y = 0$

Type : Invariance of Euler's equation

Example : Find the extremals of the functional $I\left(y(x) \right) = \int_{x_1}^{x_2} \sqrt{1 + y'^2} \, dx$.

or

Find the extremals of the functional $\int_{\theta_1}^{\theta_2} \sqrt{r^2 + r'^2} \, d\theta$ by using the transformations $x = r\cos\theta$ and $y = r\sin\theta$.

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Solution : Let $x = r\cos\theta$, $y = r\sin\theta$

$$dx = \cos\theta dr - r\sin\theta d\theta \quad \text{......(1)}$$

$$dy = \sin\theta dr + r\cos\theta d\theta \quad \text{......(2)}$$

$$\frac{dx}{d\theta} = \cos\theta\frac{dr}{d\theta} - r\sin\theta\frac{d\theta}{d\theta}$$

$$\frac{dy}{d\theta} = \sin\theta\frac{dr}{d\theta} + r\cos\theta\frac{d\theta}{d\theta}$$

$$\frac{dx}{d\theta} = \cos\theta\frac{dr}{d\theta} - r\sin\theta \quad \text{......(3)}$$

$$\frac{dy}{d\theta} = \sin\theta\frac{dr}{d\theta} + r\cos\theta \quad \text{.......(4)}$$

Squaring and adding equation (3) and (4)

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2 = r^2 + r'^2$$

$$\Rightarrow \int_{x_1}^{x_2}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta = \int_{x_1}^{x_2}\sqrt{(dx)^2 + (dy)^2} = \int_{x_1}^{x_2}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$\Rightarrow \int_{x_1}^{x_2}\sqrt{1 + y'^2}\,dx$$

Let $F(x, y, y') = \sqrt{1 + y'^2}$ $\Rightarrow$ $F_y = 0,\ F_{y'} = \dfrac{y'}{\sqrt{1 + y'^2}}$

$$0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0 \Rightarrow \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0$$

Integrating both sides $\dfrac{y'}{\sqrt{1 + y'^2}} = c$ $\Rightarrow$ $\dfrac{\sqrt{1 + y'^2}}{y'} = \dfrac{1}{c} = c$

$$\Rightarrow \sqrt{1 + y'^2} = y'c \quad \Rightarrow \quad 1 + y'^2 = y'^2c^2 \quad \Rightarrow \quad 1 = y'^2\left(c^2 - 1\right)$$

$$\Rightarrow y'^2 = \frac{1}{\left(c^2 - 1\right)} \quad \Rightarrow \quad y' = \frac{1}{\sqrt{c^2 - 1}} = c_1 \ (\text{say}) \quad \Rightarrow \quad \frac{dy}{dx} = c_1$$

$$\Rightarrow \int dy = \int c_1\,dx \quad \Rightarrow \quad y = c_1x + c_2$$

$$\Rightarrow r\sin\theta = c_1 r\cos\theta + c_2$$

Exercise 1.9

1. Find the extremals of the functional $I\left[r(\theta)\right]=\int_{\theta_1}^{\theta_2} r\sin\theta\sqrt{r^2+r'^2}\,d\theta$ by using the transformations $x=r\cos\theta$ and $y=r\sin\theta.$

2. Find the extremals of the functional $I\left[y(x)\right]=\int_{0}^{\log 2}\left(e^{-x}y'^2-e^{x}y^2\right)dx,$ by using the transformations $x=\log u$ and $y=v$, where $\log 2=\log_e 2$.

3. Find the extremals through $(0,0)$ and $(1,1)$ of $I\left[y(x)\right]=\int_{x_1}^{x_2} y^2\left(y'^2-x^2\right)dx$ by using the transformations $x^2=u$ and $y^2=v.$

Answers

1. $\log\left(r\sin\theta+\sqrt{r^2\sin^2\theta-c_1^{\,2}}\right)=c_2+\dfrac{r\cos\theta}{c_1}$ 2. $y=c_1\cos e^x+c_2\sin e^x$ 3. $y=\dfrac{7x-x^4}{6}$

Type : Applications of Euler's equation

Example : Show that the shortest distance between two points in a plane is straight line.

Solution : As we know that the shortest distance between two points in plane is

$$I\left[y(x)\right]=\int_{x_1}^{x_2}\sqrt{1+y'^2}\,dx$$

Let $F(x,y,y')=\sqrt{1+y'^2}$ $\Rightarrow$ $F_y=0,\ F_{y'}=\dfrac{y'}{\sqrt{1+y'^2}}$

By Euler equation $0-\dfrac{d}{dx}\left(\dfrac{y'}{\sqrt{1+y'^2}}\right)=0$

Integrating both sides $\dfrac{y'}{\sqrt{1+y'^2}}=c$ $\Rightarrow$ $\dfrac{\sqrt{1+y'^2}}{y'}=\dfrac{1}{c}=c$ $\Rightarrow$ $\sqrt{1+y'^2}=y'c$

$\Rightarrow$ $1+y'^2=y'^2c^2$ $\Rightarrow$ $y'^2\left(c^2-1\right)=1$ $\Rightarrow$ $y'^2=\dfrac{1}{c^2-1}$

$\Rightarrow$ $y'=\dfrac{1}{\sqrt{c^2-1}}=c_1$ $\Rightarrow$ $y'=c_1$ $\Rightarrow$ $\dfrac{dy}{dx}=c_1$

$\Rightarrow$ $\int dy=\int c_1\,dx$ $\Rightarrow$ $y=c_1x+c_2$

which is a required equation of a straight line.

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Exercise 1.10

1. Find the curve passing through the point (x_1, y_1) and (x_2, y_2) which rotated about the *x*-axis gives a minimum surface area.
2. Find the path on which a particle in the absence of friction will slide from one point to another in the shortest time under the action of gravity.

Answers

1. $y = c_1 \cosh\left(\dfrac{x + c_2}{c_1}\right)$ 2. $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$

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2. Isoperimetric Problems

Isoparametric problem : The problem of finding closed curve of the given length which in closes maximum area the called this problem as isoparametric problem.

Working Rules :

(i) Suppose we wish to find a curve y=y(x) that gives extreme value of the functional $\int_{x_1}^{x_2} f(x, y, y')\,dx$

keeping another integral $\int_{x_1}^{x_2} g(x, y, y')\,dx$ = constant.

(ii) Let $y = y(x)$ satisfy the boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$.

(iii) Let $F = f(x, y, y') + \lambda g(x, y, y')$ where λ is called Lagrange's multiplier.

(iv) Euler's equation $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$ *i.e* $fy - \dfrac{d}{dx} fy' + \lambda\left(g_y - \dfrac{d}{dx} g_{y'}\right) = 0$

(v) We have to determine λ and two constants of integration.

Example 1. Find the extremal of the functional $\int_0^2 y'^2\,dx$ under the constraint $\int_0^2 y\,dx = 1$ given $y(0) = 0$ and $y(2) = 1$.

Solution : $F = y'^2 + \lambda y,$ $y(0) = 0,\ y(2) = 1$

$$\lambda - \frac{d}{dx}(2y') = 0 \quad \Rightarrow \quad \lambda - 2y'' = 0 \quad \Rightarrow \quad y'' = \frac{\lambda}{2}$$

$$y(x) = \frac{\lambda}{4}x^2 + c_1 + c_2 x, \qquad y(0) = 0 \quad \Rightarrow \quad c_1 = 0 \quad \Rightarrow \quad y(x) = \frac{\lambda}{4}x^2 + c_2 x$$

$y(2)=1 \quad \Rightarrow \quad 1=\lambda+2c_2 \quad \Rightarrow \quad c_2=\frac{1-\lambda}{2}$

$\Rightarrow \quad y(x)=\frac{\lambda}{4}x^2+\left(\frac{1-\lambda}{2}\right)x$

$\int_0^2 y\,dx=1 \quad \Rightarrow \quad \int_0^2\left[\frac{\lambda}{4}x^2+\left(\frac{1-\lambda}{2}\right)x\right]dx=1$

$\Rightarrow \quad \left[\frac{\lambda x^3}{12}+\left(\frac{1-\lambda}{2}\right)\frac{x^2}{2}\right]_0^2=1$

$\Rightarrow \quad \frac{8\lambda}{12}+(1-\lambda)=1 \quad \Rightarrow \quad 8\lambda+12-12\lambda=12 \quad \Rightarrow \quad \lambda=0$

$\Rightarrow \quad y(x)=\frac{1}{2}x \quad =\frac{x}{2}$

Example 2. Find the plane curve of fixed length having maximum area.

Solution : Let I be the fixed parimeter of a plane curve, between points $x=x_1$ and $x=x_2$ then

$$I=\int_{x_1}^{x_2}\sqrt{1+y'^2}\,dx \qquad \ldots\ldots(1)$$

Let
$$A=\int_{x_1}^{x_2} y\,dx \qquad \ldots\ldots(2)$$

$f(x,y,y')=y,\ \ g(x,y,y')=\sqrt{1+y'^2}$

$F=f+\lambda g \quad \Rightarrow \quad F=y+\lambda\sqrt{1+y'^2}$

$\Rightarrow \quad \frac{\partial F}{\partial y}=1,\quad \frac{\partial F}{\partial y'}=\frac{\lambda y'}{\sqrt{1+y'^2}}$

$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0 \quad \Rightarrow \quad 1-\frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right)=0$

$\frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right)=1$

Now, integrating on both sides we get, $\frac{\lambda y'}{\sqrt{1+y'^2}}=x-c_1$

Squaring both sides we get, $\left(\lambda^2-(x-c_1)^2\right)y'^2=(x-c_1)^2$

$y'=\pm\frac{(x-c_1)}{\sqrt{\lambda^2-(x-c_1)^2}}$

Integrating above equation we get, $y=\pm\sqrt{\lambda^2-(x-c_1)^2}+c_2$

$\Rightarrow \quad y-c_2=\pm\sqrt{\lambda^2-(x-c_1)^2}$

$(x-c_1)^2+(y-c_2)^2=\lambda^2$ which is a circle.

Hence the curve is a circle.

Exercise 2.1

1. Find the external for which the functional $I=\int_0^1 y'^2\,dx$, $y(0)=0$, $y(1)=1$ is extremum, subject to the condition $I=\int_0^1 y\,dx=2$.

2. Find the extremal of the functional $\int_0^1 y'^2\,dx$; $y(0)=1$, $y(1)=6$ subject to the condition $\int_0^1 y\,dx=3$

3. Prove that the extemal of the functional $I[y(x)]=\int_1^4 y'^2\,dx$; $y(1)=3$, $y(4)=24$ subject to the condition $\int_1^4 y\,dx=36$ is a parabola.

4. Find the extremal of the functional $\int_{x_1}^{x_2} y'^2\,dx$ subject to the condition $\int_{x_1}^{x_2} y\,dx=c$, a constant.

5. Find the extremal of the functional $\int_0^{\pi}(y'^2-y^2)\,dx$ under the constraint $\int_0^{\pi} y\,dx=1$; $y(0)=0$, $y(\pi)=1$.

6. Find all the functions y, that satisfy the conditions $y(0)=y(\pi)=0$ and which under the integral $I=\int_0^{\pi} y'^2\,dx$ is stationary subject to the constraint $\int_0^{\pi} y^2\,dx=1$.

7. Find the extremal of the functional $\int_0^1\left(x^2+y'^2\right)dx$ under the constraint $\int_0^1 y^2\,dx=2$; $y(0)=0$, $y(1)=0$.

Answers

1. $y=-9x^2+10x$
2. $y=3x^2+2x+1$
3. $y=x^2+2x$
4. $y=\frac{3}{2}\cdot\frac{2c-c_1\left(x_2^{\,2}-x_1^{\,2}\right)-2c_2\left(x_2-x_1\right)}{x_2^{\,3}-x_1^{\,3}}x^2+c_1x+c_2$
5. $y=\frac{1}{2}(1-\cos x)+\frac{2-\pi}{4}\sin x$
6. $y=\pm\sqrt{\frac{2}{\pi}}\sin nx, n\in\mathbb{N}$
7. $y=\pm 2\sin n\pi x, n\in\mathbb{N}$

3. Variational Problems for functionals involving several dependent variables

If $I\left(y_1(x), y_2(x), ..., y_n(x)\right) = \int_{x_1}^{x_2} F\left(x, y_1(x), y_2x, ..., y_nx, y_1'(x), y_2'(x), ..., y_n'(x)\right)dx$ then

Euler's equation is: $F_{y_i} - \frac{d}{dx}F_{y_i'} = 0$ where $i = 1, 2, ..., n$

i.e. $F_{y_1} - \frac{d}{dx}F_{y_1'} = 0$, $F_{y_2} - \frac{d}{dx}F_{y_2'} = 0$,...., $F_{y_n} - \frac{d}{dx}F_{y_n'} = 0$

Example : Find the extremals of the functional $I[y(x), z(x)] = \int_0^1 \left(y'^2 + z'^2\right)dx$; $y(0) = 0,\ z(0) = 0,$ $y(1) = 1$, $z(1) = 2$

Solution : $F\left(x, y, y', z, z'\right) = y'^2 + z'^2$; $y(0) = 0 = z(0),\ y(1) = 1,\ z(1) = 2$

$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ and $\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) = 0$

$\Rightarrow$ $-\frac{d}{dx}(2y') = 0$ and $-\frac{d}{dx}(2z') = 0$

$\Rightarrow$ $y'' = 0$ and $z'' = 0$

$\Rightarrow$ $y(x) = c_1 + c_2x$ and $z(x) = c_3 + c_4x$

$y(0) = 0$ $\Rightarrow$ $c_1 = 0$ and $z(0) = 0$ $\Rightarrow$ $c_3 = 0$

$y(1) = 1$ $\Rightarrow$ $c_2 = 1$ and $z(1) = 2$ $\Rightarrow$ $c_4 = 2$

$\Rightarrow$ $y(x) = x$ and $z(x) = 2x$

$\therefore$ $y(x) = x$ and $z(x) = 2x$

Exercise 3.1

1. Find the extremals of the functional $I[y(x), z(x)] = \int_0^1 \left(y'^2 + z'^2 + 4z\right)dx$; $y(0) = 0, z(0) = 0,$ $y(1) = 1$, $z(1) = 0$

2. Show that the functional $\int_0^1 \left(2x + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)dt$ $s.t$ $x(0) = 1,\ y(0) = 1,\ x(1) = 1.5,\ y(1) = 1$ is stationary for $x(t) = \frac{2 + t^2}{2}$, $y(t) = 1.$

3. Find the extremals of the functional $I[y(x), z(x)] = \int_0^{\frac{\pi}{2}} (y'^2 + z'^2 + 2yz)dx$ satisfying $y(0) = 0,\ y\left(\frac{\pi}{2}\right) = 1,\ z(0) = 0,\ z\left(\frac{\pi}{2}\right) = -1$

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4. Find the extremals of the functional $I = \int_0^{\frac{\pi}{2}} \left(\dot{x}^2 + \dot{y}^2 + 2xy\right) dt$, such that at $t = 0$, $x = y = 0$ and at $t = \frac{\pi}{2}$, $x = y = 1$ where $\dot{x} = \frac{dx}{dt}$ and $\dot{y} = \frac{dy}{dt}$

5. Find the extremals of the functional $I[y(x), z(x)] = \int_0^{\pi} \left(2yz - 2y^2 + y'^2 - z'^2\right) dx$; $y(0) = 0$, $y(\pi) = 1$, $z(0) = 0$, $z(\pi) = 1$

6. Find the extremals of the functional $I[y(x), z(x)] = \int_0^1 \sqrt{1 + y'^2 + z'^2}\, dx$; $y(0) = 0$, $y(1) = 2$, $z(0) = 0$, $z(1) = 4$

7. Find the extremals of the functional $I = \int_0^1 \left(2\dot{x}\dot{y} + y^2 + x^2\right) dt$ s.t. at $t = 0, x = y = 1$ and at $t = 1, x = y = 0$

Answers

1. $y(x) = x$ and $z(x) = x^2 - x$

3. $y(x) = \sin x$ and $z(x) = -\sin x$.

4. $x(t) = y(t) = \text{cosec } h\left(\frac{\pi}{2}\right) \sin h t$.

5. $y(x) = c_3 \sin x - \frac{x}{\pi} \cos x$ and $z(x) = c_3 \sin x + \frac{1}{\pi}(2\sin x - x\cos x)$.

6. $z(x) = 4x$ and $y(x) = 2x$

7. $x(t) = y(t) = \frac{\sinh(1-t)}{\sinh 1}$

---

4. Functional dependent on higher order derivatives

If $I = \int_{x_1}^{x_2} F\left(x, y, y', y'', \ldots, y^{(n)}\right) dx$, then

Euler's equation is: $F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2}\left(F_{y''}\right) + \ldots + (-1)^n \frac{d^n}{dx^n}\left(F_{y^{(n)}}\right) = 0$ also known as Euler-Poisson equation.

Particular cases : If $I = \int_{x_1}^{x_2} F\left(x, y, y', y''\right) dx$ then $F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} = 0$.

If $I=\int_{x_1}^{x_2} F\left(x,y,y',y'',y'''\right)dx$ then $F_y-\frac{d}{dx}F_{y'}+\frac{d^2}{dx^2}F_{y''}-\frac{d^3}{dx^3}F_{y'''}=0$.

Example 1. Find the extremal of the functional

$I=\frac{1}{2}\int_0^1 y''^2 dx$; $y(0)=0$, $y(1)=\frac{1}{2}$, $y'(0)=0$, $y'(1)=1$.

Solution : $F\left(x,y,y'\right)=\frac{1}{2}y''^2$, $y(0)=0=y'(0)$, $y(1)=\frac{1}{2}$, $y'(1)=1$

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)=0$$

$$\Rightarrow \quad \frac{d^2}{dx}\left(\frac{1}{2}\cdot 2y''\right)=0 \quad \Rightarrow \quad y^{\mathrm{IV}}=0$$

$\Rightarrow \quad y(x)=c_1+c_2x+c_3x^2+c_4x^3$ ……..(1)

$y(0)=0 \quad \Rightarrow \quad c_1=0$ ……..(2)

$\Rightarrow \quad y(1)=\frac{1}{2} \quad \Rightarrow \quad c_2+c_3+c_4=\frac{1}{2}$ ……..(3)

$y'(x)=c_2+2c_3x+3c_4x^2$ ……..(4)

$y'(0)=0 \quad \Rightarrow \quad c_2=0$ ……..(5)

By equation (1), (2), (4) and (5) we get,

$y(x)=c_3x^2+c_4x^3$ …….(6)

$y'(x)=2c_3x+3c_4x^2$ …….(7)

Now, $y'(1)=1 \quad \Rightarrow \quad 2c_3+3c_4=1$ …….(8)

By equation (3) we get,

$c_3+c_4=\frac{1}{2}$ …….(9)

By equation (8) and (9) we get,

$c_4=0$ and $c_3=\frac{1}{2}$ …..(10)

Hence the solution is $y(x)=\frac{x^2}{2}$

Example 2. Find the extremal of the functional $I[y(x)]=\int_{x_1}^{x_2}\left(y''^2-2y'^2+y^2-2y\sin x\right)dx$

Solution : $F=y''^2-2y'^2+y^2-2y\sin x$

$$\left(2y-2\sin x\right)-\frac{d}{dx}\left(-4y'\right)+\frac{d^2}{dx^2}\left(2y''\right)=0$$

$$y-\sin x+2y''+y^{\mathrm{IV}}=0$$

$$\Rightarrow \quad y^{\mathrm{IV}}+2y''+y=\sin x$$

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$D^4+2D^2+1=\left(D^2+1\right)^2=0 \qquad \Rightarrow \qquad D^2=-1,-1$

$\Rightarrow \qquad D=\pm i,\ \pm i$

$\therefore \qquad$ C.F. $=\left(c_1+c_2x\right)\cos x+\left(c_3+c_4x\right)\sin x$

P.I. $\quad =\dfrac{1}{\left(D^2+1\right)}\left(\sin x\right)$ case fails

$=\dfrac{x}{4D^3+4D}\sin x=x\cdot\dfrac{1}{4D\left(D^2+1\right)}\sin x$ case fails

$=\dfrac{x^2}{\lfloor 2}\cdot\dfrac{1}{12D^2+4}\cdot\sin x \qquad =\dfrac{x^2}{2}\dfrac{1}{\left(-12+4\right)}\cdot\sin x=-\dfrac{x^2}{16}\sin x$

$\therefore \qquad y(x)=\left(c_1+c_2x\right)\cos x+\left(c_3+c_4x\right)\sin x-\dfrac{x^2}{16}\sin x$

Exercise 4.1

1. Find the extremal of the functional $I[y(x)]=\int_0^1\left(1+y''^2\right)dx$, $y(0)=0, y'(0)=1, y(1)=1, y'(1)=1$

2. Find the extremal of the functional $I[y(x)]=\int_{x_1}^{x_2}\left(4y^2-y''^2+2x^2\right)dx$

3. Find the extremal of the functional $I[y(x)]=\int_{x_1}^{x_2}\left(16y^2-y''^2+x^2\right)dx$

4. Find the extremal of the functional $\int_{x_1}^{x_2}\left(y''^2-y^2+x^2\right)dx$

5. Find the extremal of the functional

   $I[y(x)]=\int_0^{\frac{\pi}{2}}\left(y''^2-y^2+x^2\right)dx$; $y(0)=1,\ y'(0)=0,\ y\left(\dfrac{\pi}{2}\right)=0,\ y'\left(\dfrac{\pi}{2}\right)=-1$

6. Find the extremal of the functional

   $I[y(x)]=\int_0^{\frac{\pi}{4}}\left(y''^2-y^2+x^2\right)dx$; $y(0)=0,\ y'(0)=1,\ y\left(\dfrac{\pi}{4}\right)=y'\left(\dfrac{\pi}{4}\right)=\dfrac{1}{\sqrt{2}}$

7. Find the extremal of the functional $\int_{x_1}^{x_2}\left(360x^2y-y''^2\right)dx$

8. Find the extremal of the functional $\int_{x_1}^{x_2}\left(y^2+2y'^2+y''^2\right)dx$

9. Find the extremal of the functional $I=\int_{0}^{1}\left(y'^2+y''^2\right)dx$; $y(0)=0,\ y'(0)=1,\ y(1)=\sinh 1,\ y'(1)=\cosh 1$

10. Find the extremal of the functional $I[y(x)]=\int_{-1}^{1}\left(yy'''+\dfrac{y'''^2}{2}\right)dx$; $y(-1)=y(0)=y(1)=0$;

$y'(-1)=y'(0)=y'(1)=1$

11. Find the extremal of the functional $I[y(x)]=\int_{-1}^{0}\left(480y-y'''^2\right)dx$, $y(0)=y'(0)=y''(0)=0$

12. Find the extremal of the functional

$I=\int_{-1}^{0}\left(240y+y'''^2\right)dx$; $y(-1)=1,\ y(0)=0$; $y'(-1)=-4.5,\ y'(0)=0$; $y''(-1)=16, y''(0)=0$

13. Find the extremal of the functional $I[y(x)]=\int_{x_1}^{x_2}\left(2xy+y'''^2\right)dx$

14. Find the stationary function of the functional $\int_{x_1}^{x_2}\left(y'^2+yy''\right)dx$, subject to the conditions

$y(x_1)=\alpha,\ y'(x_1)=\beta,\ y(x_2)=\gamma,\ y'(x_2)=\delta$

15. Find the extremal of the functional $\int_{x_1}^{x_2}\left(y+\dfrac{y'''}{2}\right)dx$.

16. Find the extremal of the functional $I[y(x)]=\int_{x_1}^{x_2}\left(y'''^2+y^2-2yx^3\right)dx$

Answers

1. $y=x$

2. $y=c_1e^{x\sqrt{2}}+c_2e^{-x\sqrt{2}}+c_3\cos\left(x\sqrt{2}\right)+c_4\sin\left(x\sqrt{2}\right)$

3. $y=c_1e^{2x}+c_2e^{-2x}+c_3\cos 2x+c_4\sin 2x$

4. $y=c_1e^{x}+c_2e^{-x}+c_3\cos x+c_4\sin x$

5. $y=\cos x$.

6. $y=\sin x$

7. $y=\dfrac{x^6}{2}+c_1x^3+c_2x^2+c_3x+c_4$

8. $y=\left(c_1+c_2x\right)e^{x}+\left(c_3+c_4x\right)e^{-x}$

9. $y=\sinh x$

10. $y=x+\dfrac{3x^5-5x^3}{2}$.

11. $y=-\dfrac{x^6}{3}+c_1x^4+c_2x^5+c_3x^3$

12. $y=\dfrac{x^6}{6}+x^4+\dfrac{x^3}{6}$

13. $y=c_1+c_2x+c_3x^2+c_4x^3+c_5x^4+c_6x^6+\dfrac{x^7}{7!}$

14. No extremal

15. No extremal

16. $y=c_1e^{x}+c_2e^{-x}+e^{\frac{x}{2}}\left\{c_3\cos\left(\dfrac{x\sqrt{3}}{2}\right)+c_4\sin\left(\dfrac{x\sqrt{3}}{2}\right)\right\}+e^{\frac{-x}{2}}\left\{c_5\cos\left(\dfrac{x\sqrt{3}}{2}\right)+c_6\sin\left(\dfrac{x\sqrt{3}}{2}\right)\right\}+x^3$

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5. Functionals dependent on the functions of several independent variables

$I\left[z(x),z(y)\right]=\iint\limits_D F\left(x,y,z,z_x,z_y\right)dx\,dy$ where x,y are two independent variables , z is dependent variable depending upon x and y and z_x, z_y are first order partial derivatives of z w.r.t. x and y respectively.

Euler's Equation is: $F_z-\frac{\partial}{\partial x}\left(F_{z_x}\right)-\frac{\partial}{\partial y}\left(F_{z_y}\right)=0$

Euler's Equation for 2nd derivative is : $F_z-\frac{\partial}{\partial x}\left(F_{z_x}\right)-\frac{\partial}{\partial y}\left(F_{z_y}\right)+\frac{\partial^2}{\partial x^2}\left(F_{z_{xx}}\right)+\frac{\partial^2}{\partial y^2}\left(F_{z_{yy}}\right)+\frac{\partial^2}{\partial x^2 y}\left(F_{z_{xy}}\right)=0$

Example : $I\left[z(x,y)\right]=\iint\limits_D\left\{\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2\right\}dx\,dy$

Solution : $F=z_x{}^2-z_y{}^2$

$F_z-\frac{\partial}{\partial x}\left(F_{z_x}\right)-\frac{\partial}{\partial y}\left(F_{z_y}\right)=0 \quad \Rightarrow \quad 0-\frac{\partial}{\partial x}\left(2z_x\right)-\frac{\partial}{\partial y}\left(-2z_y\right)=0$

$\Rightarrow \quad \frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}=0$

Exercise 5.1

Obtain the Euler equation for the following functionals :

1. $I(z)=\iint\limits_D\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]dx\,dy$

2. $I(z)=\iint\limits_D\left[\left(\frac{\partial^2 z}{\partial x^2}\right)^2+\left(\frac{\partial^2 z}{\partial y^2}\right)^2+2\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2\right]dx\,dy$

Answers

1. $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$

2. $\frac{\partial^4 z}{\partial x^4}+\frac{2\partial^4 z}{\partial x^2\,\partial y^2}+\frac{\partial^4 z}{\partial y^4}=0$

6. Boundary Value Problems

Rayleigh-Ritz Method : The solution of Euler's differential equation along with boundary conditions amounts to extremising a certain definite integrals. This fact provides a technique of solving a boundary value problem approximately by assuming a trial solution satisfying the given boundary conditions and then extremising the integral whose integrand is found from the given differential equation.

To find approximate value of the boundary value problem

$$a(x)y''(x)+b(x)y'(x)+c(x)y(x)=f(x) \qquad \text{.....(1)}$$

such that $y(x_1)=y_1,\ y(x_2)=y_2$ .....(2)

then,

Step (i) : Construct a functional $I\left[y(x)\right]=\int_{x_1}^{x_2}\left[P(x)y'^2+Q(x)y^2+R(x)y\right]dx$ .....(3)

where $P(x)=e^{\int\frac{b}{a}dx}$ $\quad Q(x)=\dfrac{-c}{a}P(x)$ $\quad R(x)=\dfrac{2f(x)}{a}P(x)$

Step (ii) : Now approximate $y(x)$ using polynomial expressions which satisfies boundary conditions.

$$y=\sum_{i=0}^{n}c_i\phi_i(x) \qquad \text{.....(4)}$$

where $\phi_i(x)$ are polynomial and $\phi_i's\in C^2$. Also $\phi_0(x)$ satisfies boundary conditions and all other $\phi_i's$ vanish on boundary points and $c_i's$ need to be determined.

Step (iii) : Put value of y from equation. Now differentiate $I\left[y(x)\right]$ partially w.r.t c_i to get values of c_i. i.e., $\dfrac{\partial I}{\partial c_i}=0$, will give values of $c_i's$.

Put these values of $c_i's$ in equation (4) to get $y(x)$.

Example 1 : Solve by BVP $y''-y+x=0$ s.t. $y(0)=y(1)=0$.

Solution : $y''-y+x=0$ .....(1)

Such that $y(0)=y(1)=0$ .....(2)

Here $a=1,\ b=0,\ c=-1,\ f(x)=-x$

$$P(x)=e^{\int\frac{0}{1}.dx}=e^0=1 \qquad Q(x)=\frac{-c}{a}P(x)=\frac{-1.1}{-1}=1$$

$$R(x)=\frac{2f(x)}{a}.P(x)=\frac{2(-x)}{1}.1=-2x$$

$$I\left[y(x)\right]=\int_0^1\left[P(x)y'^2+Q(x)y^2+R(\alpha)y\right]dx$$

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$$I\left[y(x)\right]=\int_0^1\left[y'^2+y^2-2xy\right]dx \qquad \ldots(3)$$

where $F(x)=y'^2+y^2-2xy$ for which Euler's equation $\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$

$$\Rightarrow \quad 2y-2x-\frac{d}{dx}(2y')=0 \quad \Rightarrow \quad y-x-y''=0 \quad \Rightarrow \quad y''-y=-x$$

Let $y=c_o+c_1x+c_2x^2$ be the trial solution

$$y(0)=0 \quad \Rightarrow \quad c_0=0$$

$$y(1)=0 \quad \Rightarrow \quad c_1+c_2=0 \quad \Rightarrow \quad c_2=-c_1 \quad \Rightarrow \quad y=c_1x(1-x)$$

Put y in equation (3), we have

$$I\left[y(x)\right]=\int_0^1\left[\left[c_1(1-2x)\right]^2+c_1^2x^2(1-x)^2-2xc_1x(1-x)\right]dx$$

$$=\int_0^1\left[c_1^2(1-2x)^2+c_1^2x^2(1-x)^2-2c_1x^2(1-x)\right]dx$$

$$=c_1^2\int_0^1(1-2x)^2\,dx+c_1^2\int_0^1 x^2(1-x)^2\,dx-2c_1\int_0^1 x^2(1-x)\,dx$$

$$=c_1^2\int_0^1\left(1+4x^2-4x\right)dx+c_1^2\int_0^1\left(x^2+x^4-2x^3\right)dx-2c_1\int_0^1\left(x^2-x^3\right)dx$$

$$=c_1^2\left[x+\frac{4x^3}{3}-\frac{4x^2}{2}\right]_0^1+c_1^2\left[\frac{x^3}{3}+\frac{x^5}{5}-\frac{2x^4}{4}\right]_0^1-2c_1\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1$$

$$=c_1^2\left(1+\frac{4}{3}-2\right)+c_1^2\left(\frac{1}{3}+\frac{1}{5}-\frac{1}{2}\right)-2c_1\left(\frac{1}{3}-\frac{1}{4}\right)$$

$$=c_1^2\left(\frac{1}{3}+\frac{1}{30}\right)-2c_1\frac{1}{12} \qquad I=c_1^2\frac{11}{30}-c_1\frac{1}{6}$$

Now $\frac{dI}{dc_1}=0 \quad \Rightarrow \quad 2c_1\frac{11}{30}-\frac{1}{6}=0 \quad \Rightarrow \quad c_1=\frac{1}{6}\times\frac{15}{11}=\frac{5}{22}$

Thus, the approximate $y(x)=\frac{5}{22}\left(x-x^2\right)$

Example 2 : Find the approximate solution of $y''+xy=-x,\ y(0)=y(1)=0$ and $F(x,y,y')=y'^2-xy^2-2xy$.

Solution : Here $I\left[y(x)\right]=\int_0^1\left(y'^2-xy^2-2xy\right)dx$ ……(1)

Let $y=c_0+c_1x+c_2x^2$ be the trial solution

$y(0)=0 \Rightarrow c_0=0$

$y(1)=0 \Rightarrow c_1+c_2=0 \Rightarrow c_1=-c_2$

$\therefore \quad y=c_1\left(x-x^2\right)$

Put value of y in equation (1), we get

$$I=\int_0^1\left[c_1^2(1-2x)^2-c_1^2x\left(x-x^2\right)^2-2c_1x\left(x-x^2\right)\right]dx$$

$$=\int_0^1\left[c_1^2\left(1+4x^2-4x\right)-c_1^2\left(x^3+x^5-2x^4\right)-2c_1\left(x^2-x^3\right)\right]dx$$

$$=c_1^2\left(x+\frac{4x^3}{3}-\frac{4x^2}{2}\right)_0^1-c_1^2\left(\frac{x^4}{4}+\frac{x^6}{6}-\frac{2x^5}{5}\right)_0^1-2c_1\left(\frac{x^3}{3}-\frac{x^4}{4}\right)_0^1$$

$$=c_1^2\left(1+\frac{4}{3}-2\right)-c_1^2\left(\frac{1}{4}+\frac{1}{6}-\frac{2}{5}\right)-2c_1\left(\frac{1}{3}-\frac{1}{4}\right)$$

$$=c_1^2\frac{1}{3}-c_1^2\frac{1}{60}-2c_1\frac{1}{12}$$

$$I=\frac{19}{60}c_1^2-\frac{1}{6}c_1$$

Now $\dfrac{dI}{dc_1}=0 \Rightarrow 2\dfrac{19}{60}c_1-\dfrac{1}{6}=0$

$\Rightarrow c_1=\dfrac{1}{6}\times\dfrac{30}{19}=\dfrac{5}{19}$

Thus approximate solution is $y=\dfrac{5}{19}\left(x-x^2\right)$

Note : Rayleigh-Ritz method can also be used to find smallest eigen value.

Example 3 : Find the approximate smallest eigen value λ of

$y''+\lambda y=0,\; y(0)=y(1)=0$ and $F\left(x,y,y'\right)=y'^2-\lambda y^2$

Solution : Here, $I=\int_0^1\left(y'^2-\lambda y^2\right)dx$ …….(1)

Let $y(x)=c_0+c_1x+c_2x^2$ be the trial solution

$y(0)=0 \Rightarrow c_0=0$

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$y(1)=0 \quad \Rightarrow \quad c_1+c_2=0 \quad \Rightarrow \quad c_1=-c_2$

$\therefore \quad y=c_1\left(x-x^2\right)$

Put value of y in equation (1), we get

$$I=\int_0^1\left[c_1^2(1-2x)^2-\lambda c_1^2\left(x-x^2\right)^2\right]dx$$

$$=c_1^2\int_0^1\left(1+4x^2-4x-\lambda x^2-\lambda x^4+2\lambda x^5\right)dx$$

$$=c_1^2\left[x+\frac{4x^3}{3}-\frac{4x^2}{2}-\frac{\lambda x^3}{3}-\frac{\lambda x^5}{5}+\frac{2\lambda\cdot x^4}{4}\right]_0^1 \qquad =c_1^2\left[1+\frac{4}{3}-2-\frac{\lambda}{3}-\frac{\lambda}{5}+\frac{\lambda}{2}\right]$$

$$I=c_1^2\left[\frac{1}{3}-\frac{\lambda}{30}\right]$$

Now $\dfrac{dI}{dc_1}=0 \Rightarrow \quad 2c_1\left(\dfrac{1}{3}-\dfrac{\lambda}{30}\right)=0 \quad \Rightarrow \quad \dfrac{1}{3}-\dfrac{\lambda}{30}=0 \quad \Rightarrow \quad \lambda=10$

$\lambda=10$ is the least approximate eigen value of the given differential equation.

Exercise 6.1

1. Solve the BVP $y''+y=-x,$ such that $y(0)=y(1)=0$
2. Solve $y''=1, y(0)=y(1)=0$
3. Find an approximate solution for the functional $I\left[y(x)\right]=\int_0^1\left(\frac{1}{2}y'^2-y\right)dx$, s.t. $y(0)=y(1)=0$
4. Find an approximate solution for the functional $I\left[y(x)\right]=\int_0^1\left[y'^2-y^2-2xy\right]dx$ such that

   $y(0)=1,\ y(1)=2$
5. Find the least eigen value of $y''+\lambda y=0,\ y'(0)=0,\ y(1)=0$.

Answers

1. $y(x)=\frac{5}{18}\left(x-x^2\right)$ 2. $y(x)=\frac{1}{2}\left(x^2-x\right)$ 3. $y(x)=\frac{1}{2}\left(x-x^2\right)$

4. $y(x)=1+x+\frac{5}{4}\left(x-x^2\right)$ 5. least eigen value is 3.

7. Weierstrass Function

Let $I\left[y(x)\right]=\int_{x_1}^{x_2} F\left(x,y,y'\right)dx$ be a functional with fixed boundaries i.e., $y\left(x_1\right)=y_1$ and $y\left(x_2\right)=y_2$.

Then Weierstrass function is defined as $E\left(x,y,y',p\right)=f\left(x,y,y'\right)-f\left(x,y,p\right)-\left(y'-p\right)f_p\left(x,y,p\right)$

where p is the slope of the extremal. i.e., $p=\frac{dy}{dx}$.

Weierstrass function is used to check maxima /minima and strong / weak for a functional.

S. No.	Sufficient Condition	Nature of extremal of the given functional
1.	$E\ge 0,$ for arbitrary y'	strong minima
2.	$E\ge 0,$ for some y'	weak minima
3.	$E\le 0,$ for arbitrary y'	strong maxima
4.	$E\le 0,$ for some y'(close to p)	weak maxima

Example 1 : Test for an extremum the functional $I\left[y(x)\right]=\int_0^1\left(x+2y+\frac{y'^2}{2}\right)dx; y(0)=y(1)=0$

Solution : $F\left(x,y,y'\right)=x+2y+\frac{y'^2}{2}$

By Euler-Lagrange equation $\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$

$\Rightarrow \quad 2-\frac{d}{dx}\left(y'\right)=0 \qquad \Rightarrow \quad y''-2=0$

$\Rightarrow \quad y'=2x+c_1 \qquad \Rightarrow \quad y=x^2+c_1x+c_2$

Now $y(0)=0 \quad \Rightarrow \quad c_2=0, \qquad y(1)=0 \quad \Rightarrow \quad c_1=-1$

$\therefore \quad y=x^2-x$

Extremal is : $y=x^2-x \qquad p=2x-1$

Weierstrass function is :

$$E\left(x,y,y',p\right)=F\left(x,y,y'\right)-F\left(x,y,p\right)-\left(y'-p\right)F_p\left(x,y,p\right)$$

$$E=x+2y+\frac{y'^2}{2}-\left(x+2y+\frac{p^2}{2}\right)-\left(y'-p\right)p$$

$$E\left(x,y,y',p\right)=\frac{1}{2}\left(y'^2-p^2\right)-\left(y'-p\right)p$$

$$\Rightarrow \quad \frac{1}{2}\left(y'-p\right)\left(y'+p-2p\right) \qquad =\frac{\left(y'-p\right)^2}{2}$$

Now $E \geq 0$ for all y'

$\Rightarrow$ This extremal $y = x^2 - x$ will give minimum value of the functional. This minima will be strong minima.

Legendre Condition : Let $I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')dx$ be a functional with fixed boundaries $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Let $F(x, y, y')$ possess a continuous partial derivative $F_{y'y'}$ i.e., $\frac{\partial^2 F}{\partial y' \partial y'}$. Let C be the curve of extremal of given functional which passes through P_1 and P_2. Then, the following table provides the sufficient conditions for the nature of the extremal of the given functional.

S.No.	Sufficient Condition	Nature of extremal of the given functional
1.	$f_{y'y'} \geq 0$, at points close to C and for some y'	weak minima
2.	$f_{y'y'} \geq 0$, at points close to C and also for arbitrary y'	strong minima
3.	$f_{y'y'} \leq 0$, at points close to C and for some y'	weak maxima.
4.	$f_{y'y'} \leq 0$, at points close to C and for arbitrary y'	strong maxima.

Example 2 : Test for an extremum the functional $I[y(x)] = \int_0^1 e^x \left(y^2 + \frac{1}{2} y'^2 \right) dx$

Solution : $F(x, y, y') = e^x \left(y^2 + \frac{1}{2} y'^2 \right)$

Euler-Lagrange equation is $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'} \right) = 0$

$\Rightarrow \quad e^x \cdot 2y - \frac{d}{dx}(e^x \cdot y') = 0 \qquad \Rightarrow \quad e^x \cdot 2y - e^x y'' - e^x y' = 0$

$\Rightarrow \quad e^x (y - y'') = 0 \qquad \Rightarrow \quad y'' - y = 0 \qquad (e^x \neq 0)$

A.E. is $D^2 - 1 = 0$

$\Rightarrow \quad D = \pm 1 \qquad \Rightarrow \quad y = c_1 e^x + c_2 e^{-x}$

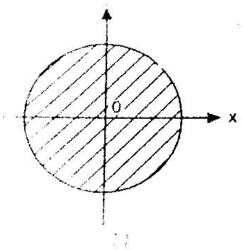
Now $F_{y'} = e^x y'$ $\qquad F_{y'y'} = e^x > 0$, for all y'

It follows that on the curve C i.e., $C : y = c_1 e^x + c_2 e^{-x}$, a strong minima is attained.

Proper field, central field and field of extremals :

Proper field : A family of curves $y = y(x, c)$ is said to form a proper field in a given region D of the xy plane if one and only one curve of the family passes through every point of the region D. For example, inside the circle $x^2 + y^2 = 1$, the family of parallel lines $y = x + c$ (c being an arbitrary constant) forms a proper field since through any point of the above circle there passes one and only

one straight line of the family. On the other hand, the family of parabolas $y=(x-c)^2-1$ inside the same circle $x^2+y^2=1$ does not form a proper field since the parabolas of this family interest inside the circle.



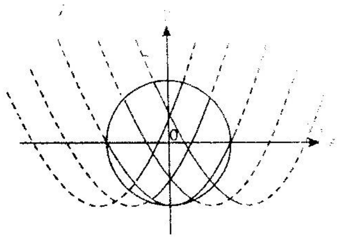


Fig. (ii)

Central Field : If all the curves of the family $y=y(x,c)$ pass through a certain point (x_0,y_0), i.e., if they form a pencil of curves, then they do not form a proper field in the region *D*, if the centre of the pencil (x_0,y_0) belongs to *D*. However, if the curves of the pencil cover the entire region *D* and do not intersect anywhere in this region, with the exception of the centre of the pencil (x_0,y_0), then the family $y=y(x,c)$ is said to form a central field.

For example, the pencil of sinusoids $y=c\sin x$ for $0\le x\le a, a<\pi$ forms a central field. But the above mentioned pencil of sinusoids forms a proper field in a sufficiently small neighbourhood of the segment of *x*-axis for $\delta\le x\le a,$ where $\delta>0,\ a\le\pi$. Again the above mentioned pencil of sinusoids does not form a proper field in a neighbourhood of the segment of *x*-axis, for $0\le x\le a_1,\ a_1>\pi$.

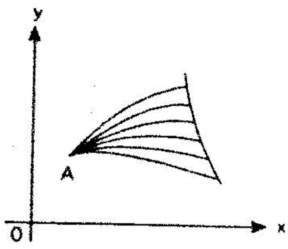


Fig. (III)

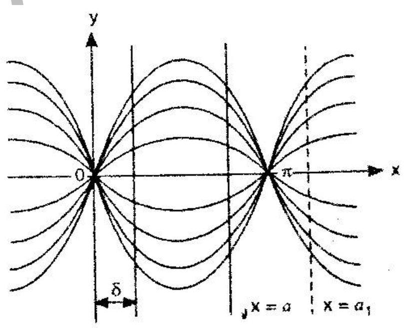


Fig. (iv)

Extremal Field : If a proper field or a central field is formed by a family of extremals of a given variational problem, then it is known as an extremals field.

Example 1. Consider the functional $I[y(x)]=\int_0^a (y'^2-y^2)\,dx$

Let it be required to include the arc of the extremal $y=0$ that connects the points $(0,0)$ and $(a,0)$ where $0<a<\pi$ in the central field of extremals.

Solution : Comparing the given functional with $\int_0^a F(x,y,y')\,dx$, we have

$$F(x,y,y')=y'^2-y^2 \qquad \text{…….(1)}$$

Euler's equation is
$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0 \qquad \text{…….(2)}$$

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From (1),

$\frac{\partial F}{\partial y} = -2y,\ \frac{\partial F}{\partial y'} = 2y'$ and

Using these values, (2) reduces to

$-2y - 2y' = 0$ or $(D^2+1)y = 0,$ where $D = d/dx$ ....(3)

The general solution of (3) is $y = c_1 \cos x + c_2 \sin x$ ....(4)

Since, the required extremal passes through $(0,0),(4)$ yields, $c_1 = 0$ and hence (4) yields.

$$y = c_2 \sin x \qquad ....(5)$$

And the curves of this pencil form as central field on the interval $0 \le x \le a,\ a < \pi$ including, for $c_2 = 0,$ the extremal is $y = 0$

From (5), $\frac{dy}{dx} = c_2 \cos x$ so that $c_2 = \left(\frac{dy}{dx}\right)_{(0,0)}$, showing that the parameter of the family of extremals (5) can be taken as the value of $\frac{dy}{dx}$ at $(0,0)$. However, in the above problem, if $a \ge \pi,$ then the family of extremals (5) does not form an extremal field.

Example 2. Find the proper and central fields of extremals for the functional $\int_0^{\frac{\pi}{4}} \left(y'^2 - y^2 + 2x^2 + 4\right) dx$

Solution : Comparing the given functional with $\int_0^{\frac{\pi}{4}} F(x,y,y')\,dx$, $F(x,y,y') = y^2 - y^2 + 2x^2 + 4$

$\therefore$ Euler's equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \Rightarrow \quad -2y - \frac{d(2y')}{dx} = 0$

$\frac{d^2y}{dx^2} + y = 0$ or $(D^2+1)y = 0,$ where $D = \frac{d}{dx}$ ....(1)

Auxiliary equation of (1) is $D^2 + 1 = 0$ so that $D = \pm 1$ and hence solution of (1) is given by

$y = c_1 \cos x + c_2 \sin x,$ c_1 and c_2 being arbitrary constants ....(2)

(2) in the equation of the extremals.

For $c_2 = 0,$ (2) yields $y = c_1 \cos x,$ which is a proper field of extremals in the domain $0 \le x \le \frac{\pi}{4}$.

Again, for $c_1 = 0,$ (2) yields $y = c_2 \sin x,$ which is a central field of extremals in the domain $0 \le x \le \frac{\pi}{4}$.

Example 3. Show that the extremal of the variational problem $\int_0^2 \left(y'^3 + \sin^2 x\right)dx, y(0) = 0,\ y(2) = 6$ is included in a central field of extremals of the given functional.

Solution : Comparing the given functional with $\int_0^2 F(x, y, y')dx,\ F(x, y, y') = y'^3 + \sin^2 x$

$\therefore$ Euler's equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \Rightarrow \quad 0 - \frac{f\left(3y'^2\right)}{dx} = 0$

Integrating it, $y'^2 = c_1^2$ or $y' = \frac{dy}{dx} = c_1,$ so that $y = c_1 x + c_2$ ....(1)

Where c_1 and c_2 are arbitrary constants.

(1) is the equation of the extremals. Using the given boundary conditions $y(0) =$ and $y(2) = 6,$ (1) gives

$0 = c_2$ and $6 = 2c_1 + c_2$ so that $c_1 = 3$ and $c_2 = 0.$

Therefore, (2), $\Rightarrow y = 3(x,)$ which is the extremal of the given variational problem.

For $c_2 = 0,$ (1) yields $y = c_1 x,$ which is a central field of extremals in the domain $0 < x \le 2$ with centre at $(0,0).$ For $c_1 = 3,$ $y = c_1 x$ reduces, to $y = 3x,$ showing that the extremal $y = 3x$ is included in the central field of extremals $y = c_1 x$.

Exercise 7.1

Test for an extremum the following functionals :

1. $I\left[y(x)\right] = \int_0^a y'^3\, dx;\ y(0) = 0, y(a) = b, a > 0, b > 0$
2. $I\left[y(x)\right] = \int_0^2 \left(e^{y'} + 3\right)dx;\ y(0) = 0,\ y(2) = 1$
3. $I\left[y(x)\right] = \int_0^1 yy'^2\, dx;\ y(0) = 4, y(1) = 4$

Answers

1. Weak minima on extremal
2. strong minima
3. strong minima

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8. Variational Problems with moving Boundaries

Let the functional $I[y(x)]=\int_{x_1}^{x_2} F(x,y,y')dx$ in which the boundary points are not specified, but are to be determined along the unknown function. i.e., x_1 and x_2 are not specified but (x_1,y_1) and (x_2,y_2) lie on the curves. Suppose the boundary point (x_2,y_2) moves along the curve $y=\psi(x)$. Then, the sufficient condition to determine the extremals on which functional attains extremum is given by

$$F+(\psi'-y')\frac{\partial F}{\partial y'}=0 \quad \text{at} \quad x=x_2$$

which is known as transversality condition.
If the boundary point (x_1,y_1) moves along the curve $y=\phi(x)$ then as above transversality condition is.

$$F+(\phi'-y')\frac{\partial F}{\partial y'}=0 \qquad \text{at } x=x_1$$

Remark : 1. If (x_2,y_2) moves along a vertical line, then transversality condition is $\left[F_{y'}\right]_{x=x_2}=0$

2. If (x_2,y_2) moves along horizontal line, then transversality condition is $\left[F-y'F_{y'}\right]_{x=x_2}=0$

Moving boundary problems : An elementary problem with moving boundaries.
Let $F=F(x,y,y')$ be a three times differentiable function of its arguments and in the *xy*-plane.
Let there be two curves $y=\phi(x)$ and $y=\psi(x)$ ……(1)
where $\phi(x)\in C^1[a,b]$ and $\psi(x)\in C^1[a,b]$.

We consider the functional $J[y]=\int_\gamma F(x,y,y')dx$ ……(2)

Defined on the smooth curves $y=y(x)$, the endpoints of which $A(x_0,y_0)$ and $B(x_1,y_1)$ lie on the given curves (1) so that $y_0=\phi(x_0)$, $y_1=\psi(x_1)$. It is required to find the extremum of the functional (2).

Theorem : Let the curve $\gamma : y=y(x)$ extremize the functional $J[y]=\int_\gamma F(x,y,y')dx$ from among all curves of the class C^1 joining two arbitrary points of two given curves $y=\phi(x)$, $y=\psi(x)$. Then the curve γ is an extremal and the transversality conditions $\left.\begin{matrix}\left[F+(\phi'-y')Fy'\right]\,|_{x=x_0}=0, \\ \left[F+(\psi'-y')Fy'\right]\,|_{x=x_1}=0\end{matrix}\right\}$ …..(3)

are fulfilled at the endpoints $A(x_0,y_0)$ and $B(x_1,y_1)$ of the curve γ.

Thus, to solve an elementary problem with moving boundaries it is necessary :

(i) To write down and solve the appropriate Euler equations. We then obtain a family of extremals $y = f(x, c_1, c_2)$ that is dependent on two parameters c_1 and c_2.

(ii) From the transversality conditions (3) and from the equations $\left.\begin{aligned} f(x_0, c_1, c_2) &= \phi(x_0), \\ f(x_0, c_1, c_2) &= \psi(x_1) \end{aligned}\right\}$ ……(4)

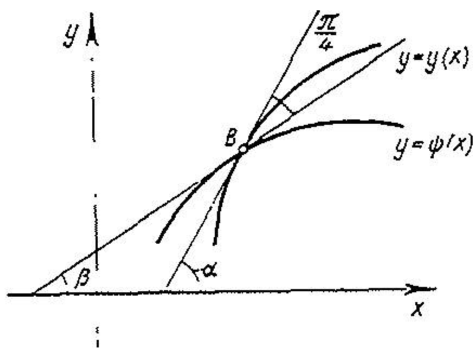
to determine the constants c_1, c_2, x_0 and x_1.

(iii) To compute the extremum of the functional (2).

Example 1. Find the transversality condition for the functional

$$J[y] = \int_{x_0}^{x_1} f(x, y) e^{\tan^{-1} y'} \sqrt{1 + y'^2}\, dx, \quad \text{where } f(x, y) \neq 0$$

Solution : Let the left end of the extremal be fixed at the



Point $A(x_0, y_0)$ and let the right end $B(x_1, y_1)$ be movable along the curve $y = \psi(x)$. We then get

$$\left[F + (\psi' - y') F_{y'}\right]\Big|_{x = x_1} = 0$$

In our case

$$F = f(x, y) e^{\tan^{-1} y'} \sqrt{1 + y'^2}$$

$$F_{y'} = f(x, y) e^{\tan^{-1} y'} \frac{1 + y'}{\sqrt{1 + y'^2}}$$

The transversality condition is :

$$\left[f(x, y) e^{\tan^{-1} y'} \sqrt{1 + y'^2} + (\psi' - y') f(x, y) e^{\tan^{-1} y'} \frac{1 + y'}{\sqrt{1 + y'^2}} \right] \Big|_{x = x_1} = 0$$

From this, by virtue of the condition $f(x, y) \neq 0$, we get

$$\frac{\psi' - y'}{1 + \psi' y'} = -1 \qquad \text{……(*)}$$

Geometrically, equation (*) signifies that the extremals $y = y(x)$ must intersect the curve $y = \psi(x)$, along which the boundary point $B(x_1, y_1)$ slides, at an angle $\frac{\pi}{4}$.

Actually, the relation (*) may be represented thus : suppose that the tangent to the extremal at the

point $B(x_1, y_1)$ lying on the curve $y = \psi(x)$ cuts the *x*-axis at an angle α, while the tangent to the given curve $y = \psi(x)$ cuts it at an angle β. Then $\tan\alpha = y'$, $\tan\beta = \psi'$ and the left member of (*) yields $\tan(\beta - x)$; but $-1 = \tan\left(-\frac{\pi}{4}\right)$; therefore $\beta - \alpha = -\frac{\pi}{4}$, whence $\alpha = \beta + \frac{\pi}{4}$, which is what we set out to prove.

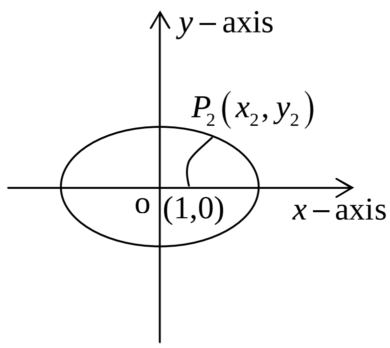
Example 2. Find the shortest distance between the point (1,0) and the ellipse

$$4x^2 + 9y^2 = 36 \qquad \text{.....(1)}$$

Solution : We have to find the shortest distance $P_1(1,0)$ and $P_2(x_2, y_2)$, where P_2 lies on the ellipse $4x^2 + 9y^2 = 36$. The arc length P_1P_2 of the minimizing curve $y = y(x)$ is given by

$$I[y(x)] = \int_1^{x_2} \left(1 + y'^2\right)^{\frac{1}{2}} dx \qquad \text{.....(2)}$$

Where the end point $P_1(1,0)$ is fixed while the end $P_2(x_2, y_2)$ lies on equation (1). Here $F = \left(1 + y'^2\right)^{\frac{1}{2}}$. Since F is independence of x and y. Euler's equation is



$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \Rightarrow \quad 0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0$$

$$\Rightarrow \frac{y'}{\sqrt{1 + y'^2}} = C \quad \Rightarrow \quad y'^2 = C^2\left(1 + y'^2\right)$$

$$\Rightarrow y'^2 = \frac{C^2}{1 - C^2} = c_1^2 \quad \Rightarrow \quad y' = c_1$$

$$\Rightarrow y = c_1 x + c_2 \qquad \text{......(3)}$$

which is a straight line along which the required shortest distance is attained.

$\therefore$ Equation (3) passes through $P_1(1,0)$

$$\Rightarrow \quad c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = -c_1$$

$\therefore$ Equation (3) becomes $y = c_1(x - 1)$ ......(4)

Also it passes through (x_2, y_2).

$$\Rightarrow \quad y_2 = c_1(x_2 - 1)$$

Now, from equation (1), $y = \frac{2}{3}\sqrt{9 - x^2} = \psi(\alpha)$ (say)

$$\psi'(x)=\frac{2}{3}\frac{(-x)}{\sqrt{9-x^2}}=\frac{-2x}{3\sqrt{9-x^2}}$$

Here, end point $P_2(x_2, y_2)$ lies on (1), i.e., $4x_2^2+9y_2^2=36$ …..(5)

Now, by using transversality condition for $\psi(x)$, we get

$$\left[F+(\psi'-y')F_{y'}\right]_{x=x_2}=0$$

$$\Rightarrow \quad \left[\sqrt{1+y'^2}+\left(\frac{-2x}{3\sqrt{9-x^2}}-y'\right)\cdot\frac{1}{2}\cdot\frac{2y'}{\sqrt{1+y'^2}}\right]_{x=x_2}=0$$

$$\Rightarrow \quad \sqrt{1+c_1^2}-\frac{2c_1x_2}{3\sqrt{1+c_1^2}\sqrt{9-x_2^2}}-\frac{c_1^2}{\sqrt{1+c_1^2}}=0 \qquad [\because \; y'=c_1]$$

$$\Rightarrow \quad \frac{1+c_1^2-c_1^2}{\sqrt{1+c_1^2}}-\frac{2c_1x_2}{3\sqrt{1+c_1^2}\sqrt{9-x_2}}=0$$

$$\Rightarrow \quad \frac{2c_1x_2}{3\sqrt{1+c_1^2}\sqrt{9-x^2}}=\frac{1}{\sqrt{1+c_1^2}} \quad \Rightarrow \quad 2c_1x_2=3\sqrt{9-x_2^2}$$

$$\Rightarrow \quad 4c_1^2x_2^2=9\left(9-x_2^2\right) \qquad \text{……(6)}$$

Now, from equation (4) and (5), we get

$$4x_2^2+9c_2^2(x_2-1)^2=36$$

$$\Rightarrow \quad 9c_1^2(x_2-1)^2=36-4x_2^2$$

$$\Rightarrow \quad 9c_1^2(x_2-1)^2=4\left(9-x_2^2\right) \qquad \text{……(7)}$$

Dividing equation (6) and (7), we get

$$\frac{4x_2^2}{9(x_2-1)^2}=\frac{9}{4} \quad \Rightarrow \quad \frac{x_2}{x_2-1}=\frac{9}{4} \quad \Rightarrow \quad x_2=\frac{9}{5}$$

From equation (6), we get

$$4c_1^2\frac{81}{25}=9\left(9-\frac{81}{25}\right) \quad \Rightarrow \; c_1^2=4 \quad \Rightarrow \quad c_1=2$$

From equation (4), we get

$$y_2=2\left(\frac{9}{5}-1\right)=\frac{8}{5} \quad \Rightarrow \quad P_2(x_2,y_2)\equiv P_2\left(\frac{9}{5},\frac{8}{5}\right)$$

$\therefore$ Required shortest distance is $=\sqrt{\left(\frac{9}{5}-1\right)^2+\left(\frac{8}{5}-0\right)^2}=\sqrt{\frac{16}{25}+\frac{64}{25}}=\frac{4\sqrt{5}}{5}$

Result : For the functional of the form $J(y)=\int_{x_0}^{x_1}h(x,y)\sqrt{1+(y')^2}\,dx$ where $h(x,y)\neq 0$ at the boundary points, the transversality conditions are of the form $y'(x)=-\frac{1}{\phi'(x)}$ and $y'(x)=-\frac{1}{\psi'(x)}$.

That is, the transversality conditions reduced to orthogonality conditions.

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Exercise 8.1

1. Find the shortest distance between the parabola $y = x^2$ and the straight line $y = x - 5$.
2. Find the shortest distance from the point $A(-1,5)$ to the parabola $y^2 = x$.
3. Find the shortest distance between the circle $x^2 + y^2 = 1$ and the straight line $x + y = 4$.
4. Find the shortest distance between the point $A(-1,3)$ and the straight line $y = 1 - 3x$.
5. Find the shortest distance between $(0,1)$ and $y = x^2$.
6. Find the shortest distance between $(0,0)$ and $x + y = \sqrt{2}$.
7. Find the shortest distance between $y = x^2$ and $x - y = 3$.

Answer

1. $\frac{19\sqrt{2}}{8}$ 2. $\sqrt{20}$ 3. $2\sqrt{2} - 1$ 4. $\frac{1}{\sqrt{10}}$ 5. $\frac{\sqrt{3}}{2}$ 6. 1 7. $\frac{11}{4\sqrt{2}}$

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---------------- S C Q ----------------

1. The functional

$$\int_0^1 \left[y'^2 + (y+2y')y'' + kxyy' + y^2 \right] dx,$$

$y(0)=0, y(1)=1, y'(0)=2, y'(1)=3$ is path independent, if k equals

1. 1   2. 2
3. 3   4. 4

(GATE 2012)

2. Assume F to be a twice continuously differentiable function. Let $J(y)$ be a functional of the form

$\int_0^1 F(x,y')dx, 0 \le x \le 1$ defined on the set of all continuously differentiable functions y on $[0,1]$ satisfying $y(0)=a, y(1)=b$. For some arbitrary constant C, a necessary condition for y to be an extremum of J is

1. $\frac{\partial F}{\partial x} = C$ 2. $\frac{\partial F}{\partial y'} = C$
3. $\frac{\partial F}{\partial y} = C$ 4. $\frac{\partial F}{\partial x} = 0$

(GATE 2011)

3. The Euler's equation for the variational problem Minimize

$I[y(x)] = \int_0^1 (2x - xy - y')y'dx,$ is

1. $2y'' - y = 2$ 2. $2y'' + y = 2$
3. $y'' + 2y = 0$ 4. $2y'' - y = 0$

(GATE 2010)

4. The extermum for the variational problem

$$\int_0^{\frac{\pi}{8}} \left[(y')^2 + 2yy' - 16y^2 \right] dx,$$

$y(0)=0, y\left(\frac{\pi}{8}\right)=1,$

occurs for the curve

1. $y = \sin(4x)$ 2. $y = \sqrt{2}\sin(2x)$
3. $y = 1 - \cos(4x)$ 4. $y = \frac{1-\cos(8x)}{2}$

(GATE 2007)

5. Let I be the functional defined by

$$I[y(x)] = \int_0^{\frac{\pi}{2}} \left\{ \left(\frac{dy}{dx}\right)^2 - y^2 \right\} dx;$$

$y(0)=0; y\left(\frac{\pi}{2}\right)=1$ where, the unknown function $y(x)$ possesses two derivatives everywhere in $\left(0, \frac{\pi}{2}\right)$. Then,

1. the functional has an extremum which cannot be achieved in the class of continuous functions.
2. the corresponding Euler's equation have a unique solution satisfying the given boundary conditions.
3. I is not linear.
4. I is linear.

(GATE 2006)

6. Extremals for the variational problem

$$v[y(x)] = \int_1^2 \left(y^2 + x^2 y'^2\right) dx$$ satisfy the

differential equation

1. $x^2 y'' + 2xy' - y = 0$
2. $x^2 y'' - 2xy' + y = 0$
3. $2xy' - y = 0$
4. $x^2 y'' - y = 0$

(GATE 2004)

7. Extremals $y = y(x)$ for the variational

problem $v[y(x)] = \int_0^1 (y + y')^2 \, dx$ satisfy

the differential equation

1. $y'' + y = 0$ 2. $y'' - y = 0$
3. $y'' + y' = 0$ 4. $y' + y = 0$

(GATE 2003)

8. The functional $\int_0^1 \left(y'^2 + 4y^2 + 8ye^x\right) dx$

$y(0) = -\frac{4}{3},\ y(1) = -\frac{4e}{3}$

possesses

1. strong minima on $y = -\frac{1}{3}e^x$
2. strong minima on $y = -\frac{4}{3}e^x$
3. weak maxima on $y = -\frac{1}{3}e^x$
4. strong maxima on $y = -\frac{4}{3}e^x$

(GATE 2012)

9. On the interval [0, 1], let y be a twice continuously differentiable function which is an extremal of the functional

$$J(y) = \int_0^1 \frac{\sqrt{1 + y'^2}}{x} dx$$ with

$y(0) = 1, y(1) = 2$. Then, for some arbitrary constant c, y satisfies

1. $y'^2\left(2 - c^2x^2\right) = c^2x^2$
2. $y'^2\left(2 + c^2x^2\right) = c^2x^2$
3. $y'^2\left(1 - c^2x^2\right) = c^2x^2$
4. $y'^2\left(1 + c^2x^2\right) = c^2x^2$

(GATE 2011)

10. The extremal of the functional

$$\int_0^1 \left(y + x^2 + \frac{y'^2}{4}\right) dx, \quad y(0) = 0, y(1) = 0$$ is

1. $4\left(x^2 - x\right)$ 2. $3\left(x^2 - x\right)$
3. $2\left(x^2 - x\right)$ 4. $x^2 - x$

(GATE 2009)

11. The possible values of α for which the variational problem,

$$J[y(x)] = \int_0^1 \left(3y^2 + 2x^3 y'\right) dx, y(\alpha) = 1$$ has

extremals are

1. $-1, 0$ 2. 0, 1
3. –1, 1 4. –1, 0, 1

(GATE 2008)

12. The functional $\int_0^1 \left(y'^2 + x^3\right) dx$, given

$y(1) = 1$, achieves its

1. weak maximum on all its extremals.
2. strong minimum on all its extremals.
3. weak maximum on some but not on all

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its extremals.

4. strong minimum on some but not on all its extremals.

(GATE 2008)

13. The functional $\int_0^1 (1+x)(y')^2\,dx, y(0)=0,$ $y(1)=1$, possesses
1. strong maxima
2. strong minima
3. weak maxima but not a strong maxima
4. weak minima but not a strong minima

(GATE 2007)

14. Let $I[y(x)]=\int_0^1 F\left(x,y,\frac{dy}{dx}\right)dx,$ satisfying $y(0)=0, y(1)=1$ where, F has continuous second order derivatives with respect to its arguments and the unknown function $y(x)$ possess two derivatives everywhere in (0, 1). If the function F depends only on x and $\frac{dy}{dx}$, then the Euler's equation is an ordinary differential equation in y which, in general, is
1. first order linear
2. first order non-linear
3. second order linear
4. second order non-linear

(GATE 2006)

15. The functional $I[y(x)]=\int_0^1\left(y+\frac{d^2y}{dx^2}\right)dx,$ defined on the set of functions C^2 [0, 1] satisfying $y(0)=1, y(1)=1, \left(\frac{dy}{dx}\right)_{x=0}=0$ and $\left(\frac{dy}{dx}\right)_{x=1}=-1$
1. only one extremal
2. exactly two extremals
3. infinite number of extremals
4. no extremals

(GATE 2006)

16. The extremum of the functional $I=\int_0^1\left[\left(\frac{dy}{dx}\right)^2+12xy\right]dx$ satisfying the conditions $y(0)=0$ and $y(1)=1$ is attained on the curve
1. $y=\sin^2\frac{\pi x}{2}$
2. $y=\sin\frac{\pi x}{2}$
3. $y=x^3$
4. $y=\frac{1}{2}\left[x^3+\sin\frac{\pi x}{2}\right]$

(GATE 2005)

17. The extremals for the functional $v[y(x)]=\int_{x_0}^{x_1}\left(xy'+y'^2\right)dx$ are given by the following family of curves
1. $y=C_1+C_2x+\left(\frac{x^2}{4}\right)$

2. $y = 1 + C_1 x + C_2\left(\frac{x^2}{4}\right)$

3. $y = C_1 + x + C_2\left(\frac{x^2}{4}\right)$

4. $y = C_1 + C_2 x - \left(\frac{x^2}{4}\right)$

(GATE 2004)

18. The functional

$$v\left[y(x)\right] = \int_0^2 \left[(y')^2 + 6xy + x^3\right] dx,$$

$y(0) = 0, y(2) = 2$ can be extremized on the curve

1. $y = x$
2. $2y = x^3$
3. $y = x^3 - 6x$
4. $2y = x^3 - 2x$

(GATE 2003)

19. An extremal of the functional

$$I\left[y(x)\right] = \int_a^b F\left(x, y, \frac{dy}{dx}\right) dx;$$

$y(a) = y_1,\ y(b) = y_2$ satisfies Euler's equation, which is general

1. is a second order linear ordinary differential equation (ODE).
2. is a non-linear ODE of order greater than two.
3. admits a unique solution satisfying the conditions $y(a) = y_1, y(b) = y_2$.
4. may not admits a solution satisfying the conditions $y(a) = y_1,\ y(b) = y_2$.

(GATE 2002)

20. The minimizing curve must satisfy a differential equation, called as

1. Lagrange's equation
2. Euler-Lagrange equation
3. Gauss equation
4. None of the above

21. If P and Q be two points in the xy-plane and P and Q be two points and $y(x)$ be a curve with $P = (a, y(a))$ and $Q = (b, y(b))$. The arc length of the curve $y(x)$ is given by the integral

1. $\int_a^b \sqrt{(1 + y(x))^2}\, dx$
2. $\int_a^b \sqrt{1 + \left[y'(x)\right]^2}\, dx$
3. $\int_a^b 1 + \left[y'(x)\right]^2 dx$
4. $\int_a^b \sqrt{1 + \left[y(x)\right]^2}\, dx$

22. Rayleigh-Ritz method is used to

1. find maxima
2. find minima
3. solve boundary value problems
4. None of the above

23. The necessary condition for the integral $\int_{x_1}^{x_2} H\, dx$ to be an extremum is

1. $\frac{\partial H}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial y'}\right) = 0$
2. $\frac{\partial H}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial \dot{x}}\right) = 0$
3. $\frac{\partial H}{\partial y'} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial y}\right) = 0$
4. $\frac{\partial H}{\partial y'} - \frac{d}{dx}\left(\frac{\partial H}{\partial \dot{x}}\right) = 0$

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24. Necessary condition for

$I=\int_{t_1}^{t_2} F(t,x,\dot{x},\ddot{x})\,dt$ to be an extremum is

that

1. $\frac{\partial f}{\partial x}-\frac{d}{dx}\left(\frac{\partial F}{\partial \dot{x}}\right)=0$

2. $\frac{\partial f}{\partial x}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)+\frac{d^2}{dt^2}\left(\frac{\partial F}{\partial \ddot{x}}\right)=0$

3. $\frac{\partial f}{\partial \dot{x}}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)+\frac{d^2}{dt^2}\left(\frac{\partial F}{\partial \dot{x}}\right)=0$

4. $\frac{\partial f}{\partial x}-\frac{\partial f}{\partial t}\left(\frac{\partial F}{\partial \dot{x}}\right)=0$

25. The extremals of the functional

$\int_{-a}^{a}\left(\lambda y+\frac{1}{2}\mu y''^2\right)dx$ which satisfies the

boundary conditions $y(-a)=0$,

$y'(-a)=0,\ y(a)=0,\ y'(a)=0$ is

1. $y=-\frac{\lambda}{\mu}\left(x^2-a^2\right)$

2. $y=-\frac{\lambda}{24\mu}\left(x^2-a^2\right)^2$

3. $y=\frac{\lambda}{24\mu}\left(a^2-x^2\right)^2$

4. $y=\frac{-\lambda}{24\mu}\left(x^2-a^2\right)$

26. Equation of the curve which gives minimum surface of revoulution about any axis say y-axis is

1. $y=c\cosh\frac{x-b}{c}$

2. $x=c\cosh\frac{y-b}{c}$

3. $y=x\cosh\frac{x-a}{b}$

4. $x=y\cosh\frac{y-a}{b}$

27. In the equation $H=f+\lambda g$, where $\int_1^2 H\,dx$

to be an extremum, λ is called as

1. Isoperimetric constant
2. Kernel's
3. Lagranges multiplier
4. Green's function

28. Extremal of the isoperimetric problem

$v[y(x)]=\int_1^4 y'^2\,dx,\ y(1)=3,\ y(4)=24$

subject to the condition $\int_1^4 y\,dx=36$ is

1. a parabola
2. straight line
3. a circle
4. a hyperbola

29. Function $y(x)$ for which $\int_0^1\left(x^2+y'^2\right)dx$ is

stationary ; given that $\int_0^1 y^2\,dx=2$;

$y(0)=0\quad y(1)=0$ is

1. $y=\sin m\pi x$
2. $y=\pm 2\sin m\pi x$
3. $y=4\sin m\pi x$
4. $y=\pm 3\sin m\pi x$

30. Curve on which the functional

$\int_0^{\frac{\pi}{2}} \left(y'^2 - y^2 + 2xy \right) dy$ with $y(0) = 0$ and $y\left(\frac{\pi}{2}\right) = 0$, be extremized is

1. $y = \frac{\pi}{2}\sin x$
2. $y = x - \frac{\pi}{2}\sin x$
3. $y = x^2 - \frac{\pi}{2}\sin x$
4. $y = -\frac{\pi}{2}\sin x$

31. Solution of boundary value problem $y'' = 3x + 4y;\; y(0) = 0, y(1) = 1$ is
1. $\bar{y} = \frac{x}{2}(5x - 4)$
2. $\bar{y} = \frac{x}{4}(5x - 1)$
3. $\bar{y} = \frac{x}{3}(5x - 1)$
4. $\bar{y} = \frac{x}{4}(5x + 1)$

32. A function $y(x)$ such that $\int_0^{\pi} y^2\, dx = 1$ which makes $\int_0^{\pi} y''^2\, dx$ a minimum, if $y(0) = 0 = y(\pi),\; y''(0) = 0 = y''(\pi)$ is
1. $y = a_n \cos nx,\; n = 0,1,2,3,...$
2. $y = a_n \sin mx,\; m = 0,1,2,3,....$
3. $y = a_n \sin nx,\; n = 0,1,2,3,....$
4. $y = \frac{a_n}{\sin nx}, n = 0,1,2,3,...$

33. Equation $\frac{d}{dx}\left\{ f - y'\frac{\partial f}{\partial y'} \right\} - \frac{\partial f}{\partial x}$ is
1. Hamilton's equation
2. Euler's equation
3. Liouville's equation
4. Bessel's equation

34. The variational problem of extremizing the functional $I(y(x)) = \int_0^{2\pi} \left[\left(\frac{d}{dx} y \right)^2 - y^2 \right] dx$; $y(0) = 1,\; y(2\pi) = 1$ has
1. a unique solution
2. exactly two solutions
3. an infinite number of solutions
4. no solution

(CSIR NET June 2011)

35. The variational problem of extremizing the functional $I(y(x)) = \int_1^3 y(3x - y)\, dx$; $y(3) = 4\frac{1}{2},\; y(1) = 1$ has
1. a unique solution
2. exactly two solutions
3. an infinite number of solutions
4. no solution

(CSIR NET June 2012)

36. Let $J(u) = \int_0^1 \left[u_x^2 + 4\frac{u^2}{x^2} \right] x dx$, where $u(x)$ is a smooth function defined on [0, 1] satisfying $u(0) = 0$ and $u(1) = 1$. Which of the functions minimizes J?
1. $u(x) = x^2$
2. $u(x) = \frac{1}{\sqrt{2}} x^2$
3. $u(x) = \frac{1}{2} x^2$

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4. $u(x)=\frac{1}{4}x^2$

(CSIR NET Dec 2012)

37. Consider the functional

$J(y)=\int_a^b F(x,y,y')dx$ where

$F(x,y,y')=y'+y$ for admissible functions

y. Then J has

1. no extremals.
2. several extemals.
3. $y(x)=e^{-x}$ as an extremal.
4. $y(x)=$ constant as an extremal.

(CSIR NET Dec 2013)

38. The curve extremizing the functional

$I(y)=\int_1^2 \frac{\sqrt{1+(y'(x))^2}}{x}dx \quad y(1)=0,\ y(2)=1$ is

1. an ellipse
2. a parabola
3. a circle
4. a straight line

(CSIR NET June 2014)

39. Consider the functional

$J(y)=y^2(1)+\int_0^1 y'^2(x)dx,\ y(0)=1$

Where $y\in C^2([0,1])$. If y extremizes J then

1. $y(x)=1-\frac{1}{2}x^2$
2. $y(x)=1-\frac{1}{2}x$
3. $y(x)=1+\frac{1}{2}x$
4. $y(x)=1+\frac{1}{2}x^2$

(CSIR NET Dec 2014)

40. The functional

$I(y(x))=\int_a^b (y^2+y'^2-2y\sin x)dx$, has the following extremal with c_1 and c_2 as arbitrary constants.

1. $y=C_1e^{2x}+C_2e^{-2x}+\frac{1}{2}\sin x$
2. $y=C_1e^{x}+C_2e^{-x}+\frac{1}{2}\sin x$
3. $y=C_1e^{x}+C_2e^{-x}-\frac{1}{2}\sin x$
4. $y=C_1e^{2x}+C_2e^{-2x}+\frac{1}{2}\cos x$

(CSIR NET Dec 2015)

41. The curve of fixed length l, that joins the points (0, 0) and (1, 0), lies above the x-axis, and encloses the maximum area between itself and the x-axis, is a segment of

1. a straight line.
2. a parabola.
3. an ellipse.
4. a circle.

(CSIR NET June 2016)

42. If $J[y]=\int_1^2 (y'^2+2yy'+y^2)dx,\ y(1)=1$ and $y(2)$ is arbitrary then the extremal is

1. e^{x-1}
2. e^{x+1}
3. e^{1-x}
4. e^{-x-1}

(CSIR NET Dec 2016)

43. The infimum of $\int_0^1 (u'(t))^2 dt$ on the class of functions $\{u \in C^1[0,1]$ such that $u(0) = 0$ and $\max_{[0,1]} |u| = 1\}$ is equal to

1. 0
2. $\frac{1}{2}$
3. 1
4. 2

(CSIR NET June 2017)

44. Let $X = \{u \in C^1[0,1] u(0) = u(1) = 0\}$ and define $J : X \to \mathbb{R}$ by $J(u) = \int_0^1 e^{-u'(x)^2} dx$.

1. J does not attain its infimum
2. J attains its infimum at a unique $u \in X$
3. J attains its infimum at exactly two elements $u \in X$
4. J attains its infimum at infinitely many $u \in X$

(CSIR NET Dec 2017)

45. Consider $J[y] = \int_0^1 \left[(y')^2 + 2y\right] dx$ subject to $y(0) = 0, y(1) = 1$. Then $\inf J[y]$

1. is $\frac{23}{12}$
2. is $\frac{21}{24}$
3. is $\frac{18}{25}$
4. does not exist

(CSIR NET June 2018)

---------------- M C Q ----------------

1. A necessary condition for $I[y(x)] = \int_{x_1}^{x_2} f(x, y, y') dx$ to be an extremum is/are

1. $\frac{\partial f}{\partial x} - \frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right) = 0$
2. $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$
3. $\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$
4. $\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x \partial y'} - y'\frac{\partial^2 f}{\partial y \partial y'} - y''\frac{\partial^2 f}{\partial y'^2} = 0$

2. Consider the functional $v[y(x)] = \int_{x_1}^{x_2} \frac{(1+y'^2)^{\frac{1}{2}}}{x} dx,$ then following/s is/are valid ; if $F = \frac{(1+y'^2)^{\frac{1}{2}}}{x}$

1. $\frac{\partial F}{\partial y'} =$ constant
2. $F - y'\frac{\partial F}{\partial y'} =$ constant
3. extremal of the given functional is a circle
4. extremal is a straight line

3. Consider the functional $I = \int_0^1 \left(2x + x'^2 + y'^2\right) dt$ such that $x(0) = 1, y(0) = 1, x(1) = 15, y(1) = 1$. Assuming $F = 2x + x'^2 + y'^2$. Then,

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1. for extremum of I

$\frac{\partial F}{\partial x}-\frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right)=0$ and

$\frac{\partial F}{\partial y}-\frac{d}{dt}\left(\frac{\partial F}{\partial y'}\right)=0$

2. $x''=1,\ y''=0$
3. $x''=0,\ y''=1$
4. $x=1+\frac{t^2}{2}, y=1$

4. Let $I=\int_0^1\left(y'^2-y^2\right)dx$. Then, select the correct option/s
   1. if $y(0)=0,\ y(1)=0,$ then extremal is $y(x)=\sin x$
   2. if $y(0)=0,$ then extremal is $y(x)=1$
   3. if $y(0)=0$, $y(1)=1,$ then extremal is $y(x)=\frac{\sin x}{\sin 1}$
   4. extremal is $y(x)=0,$ if end $y(0)=0$ and $y(1)=1$ conditions are not specified.

5. The functional

$$I[y(x)]=\int_0^1\left(x+2y+\frac{1}{2}y'^2\right)dx$$

$y(0)=0, y(1)=0$ has
   1. Weak minimum on the function $y=x^2-x$
   2. Strong minimum on the function $y=x^2-x$
   3. Weak maximum on the function $y=x^2-x$
   4. Strong maximum on the function $y=x^2-x$

6. The weak minimum of the functional

$$I[y(x)]=\int_1^2\frac{x^3}{y'^2}dx,\ \ y(1)=1,\ y(2)=4$$

attained on the curve

1. $y=x^3$ 2. $y=x^2$
3. $y=x^3-x$ 4. None of these

7. The functional $\int_0^1\left(y'^2+x^2\right)dx,\ y(1)=1$ achieves its
   1. weak maximum on all its extremals
   2. strong minimum on all its extremals
   3. weak maximum on some, but not on all of its extemals.
   4. strong minimum on some, but not on all of its extremals.

8. Consider the functional $J=\int_a^b F(x,y,y')dx$ where $F(x,y,y')=\left(1+y^2\right)/y'^2$ for admissible functions $y(x)$. Which of the following are extremals for J ?
   1. $y(x)=A\sin(x)$
   2. $y(x)=A\sinh(x)+B\cosh(x)$
   3. $y(x)=A\sinh(Ax+B)$

4. $y(x) = A\sin(x) + B\cos(x)$

(CSIR NET Dec 2012)

9. The extremal of $\int_1^2 \frac{\dot{x}^2}{t^3} dt;\ \ x(1) = 3,\ x(2) = 18$ (where $\dot{x} \equiv \frac{dx}{dt}$) using Lagrange's equation is given by which of the following ?
   1. $x = t^4 + 2$
   2. $x = \frac{15}{7}t^3 + \frac{6}{7}$
   3. $x = 5t^2 - 2$
   4. $x = 5t^3 + 3$

(CSIR NET June 2013)

10. Let $y \in C^2([0,\pi])$ satisfying $y(0) = y(\pi) = 0$ and $\int_0^\pi y^2(x)dx = 1$ extremize the functional $J(y) = \int_0^\pi (y'(x))^2 dx;\ y' = \frac{dy}{dx}$. Then
    1. $y(x) = \sqrt{\frac{2}{\pi}}\sin x$
    2. $y(x) = -\sqrt{\frac{2}{\pi}}\sin x$
    3. $y(x) = \sqrt{\frac{2}{\pi}}\cos x$
    4. $y(x) = -\sqrt{\frac{2}{\pi}}\cos x$

(CSIR NET Dec 2014)

11. The extremal of the functional $I = \int_0^{x_1} y^2 (y')^2 dx$ that passes through (0, 0) and (x_1, y_1) is
    1. a constant function
    2. a linear function of *x*
    3. part of a parabola
    4. part of an ellipse

(CSIR NET June 2015)

12. Let $y = y(x)$ be the extremal of the functional $I[y(x)] = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$, subject to the condition that the left end of the extremal moves along $y = x^2$, while the right end moves along $x - y = 5$.
    Then the
    1. shortest distance between the parabola and the straight line is $\left(\frac{19\sqrt{2}}{8}\right)$.
    2. slope of the extremal at (x, y) is $\left(-\frac{3}{2}\right)$.
    3. point $\left(\frac{3}{4}, 0\right)$ lies on the extremal.
    4. extremal is orthogonal to the curve $y = \frac{x}{2}$.

(CSIR NET June 2016)

13. Let $y = y(x)$ be the extremal of the functional $I[y(x)] = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$, subject to the condition that the left end of the extremal moves along $y = x^2$, while the right end moves along $x - y = 5$.
    Then the
    1. shortest distance between the parabola and the straight line is $\left(\frac{19\sqrt{2}}{8}\right)$.
    2. slope of the extremal at (x, y) is $\left(-\frac{3}{2}\right)$.
    3. point $\left(\frac{3}{4}, 0\right)$ lies on the extremal.

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4. extremal is orthogonal to the curve $y = \frac{x}{2}$. **(CSIR NET June 2016)**

14. The functional $J[y] = \int_0^1 (y'^2 + x^2)\,dx$ where $y(0) = -1$ and $y(1) = 1$ on $y = 2x - 1$, has
    1. weak minimum
    2. weak maximum
    3. strong minimum
    4. strong maximum

(CSIR NET Dec 2016)

15. Let $y(x)$ be a piecewise continuously differentiable function on [0, 4]. Then the functional $J[y] = \int_0^4 (y'-1)^2 (y'+1)^2\,dx$ attains minimum if $y = y(x)$ is
    1. $y = \frac{x}{2} \quad 0 \le x \le 4$
    2. $y = \begin{cases} -x & 0 \le x \le 1 \\ x - 2 & 1 \le x \le 4 \end{cases}$
    3. $y = \begin{cases} 2x, & 0 \le x \le 2 \\ -x + 6, & 2 \le x \le 4 \end{cases}$
    4. $y = \begin{cases} x, & 0 \le x \le 3 \\ -x + 6, & 3 \le x \le 4 \end{cases}$

(CSIR NET Dec 2016)

16. Consider the functional
$$I(y(x)) = \int_{x_0}^{x_1} f(x,y)\sqrt{1+y'^2}\, e^{\tan^{-1} y'}\,dx$$
where $f(x,y) \ne 0$. Let the left end of the extremal be fixed at the point $A(x_0, y_0)$ and the right end $B(x_1, y_1)$ be movable along the curve $y = \psi(x)$. Then the extremal $y = y(x)$ intersects the curve $y = \psi(x)$ along which the boundary point $B(x_1, y_1)$ slides at an angle
    1. $\frac{\pi}{3}$
    2. $\frac{\pi}{2}$
    3. $\frac{\pi}{4}$
    4. $\frac{\pi}{6}$

(CSIR NET June 2017)

17. Let $I : C^1[0,1] \to \mathbb{R}$ be defined as
$$I(u) := \frac{1}{2}\int_0^1 \left(u'(t)^2 - 4\pi^2 u(t)^2\,dt\right)$$
Let us set
$(P)\, m := \inf\left\{I(u) : u \in C^1[0,1] : u(0) = u(1) = 0\right\}$
Let $\bar{u} \in C^1[0,1]$ satisfy the Euler-Lagrange Equation associated with (P). Then
    1. $m = -\infty$ i.e. I is not bounded below
    2. $m \in \mathbb{R}$, with $I(\bar{u}) = m$
    3. $m \in \mathbb{R}$, with $I(\bar{u}) > m$
    4. $m \in \mathbb{R}$, with $I(\bar{u}) < m$

(CSIR NET Dec 2017)

18. Let $X = \left\{u \in C^1[0,1] \mid u(0) = 0\right\}$ and let $I : X \to \mathbb{R}$ be defined as
$$I(u) = \int_0^1 \left(u'(t)^2 - u(t)^2\right)dt$$
Which of the following are correct ?

1. I is bounded below
2. I is not bounded below
3. I attains its infimum
4. I does not attain its infimum

(CSIR NET Dec 2017)

19. The admissible extremal for

$$J[y]=\int_0^{\log 3}\left[e^{-x}y'^2+2e^x(y'+y)\right]dx$$

where $y(\log 3)=1$ and $y(0)$ is free is

1. $4-e^x$
2. $10-e^{2x}$
3. e^x-2
4. $e^{2x}-8$

(CSIR NET June 2018)

20. The extremal of the functional

$J[y]=\int_0^1 y'^2(x)\,dx$ subject to

$y(0)=0,\ y(1)=1$ and $\int_0^1 y(x)\,dx=0$ is

1. $3x^2-2x$
2. $8x^3-9x^2+2x$
3. $\frac{5}{3}x^4-\frac{2}{3}x$
4. $\frac{-21}{2}x^5+10x^4+4x^3-\frac{5}{2}x$

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SCQ

1. 2
2. 2
3. 2
4. 1
5. 2
6. 1
7. 2
8. 2
9. 3
10. 4
11. 3
12. 2
13. 2
14. 1
15. 4
16. 3
17. 4
18. 4
19. 4
20. 2
21. 2
22. 3
23. 1
24. 2
25. 2
26. 2
27. 3
28. 1
29. 2
30. 2
31. 2
32. 3
33. 2
34. 3
35. 4
36. 1
37. 1
38. 3
39. 2
40. 2
41. 4
42. 3
43. 3
44. 2
45. 1

MCQ

1. 2,3,4
2. 1,3
3. 1,3
4. 3
5. 1,2
6. 2
7. 2
8. 1,2,3,4
9. 1
10. 1,2
11. 3
12. 1,3
13. 1,3
14. 1,3 or 3
15. 2,4
16. 3
17. 2
18. 1,3
19. 1,3
20. 1